a. $\neg\neg\neg\phi \Rightarrow \neg\phi$ *and* $\neg\phi \Rightarrow \neg\neg\neg\phi$

b. $\neg\neg(\phi \to \psi) \Rightarrow (\neg\neg\phi \to \neg\neg\psi)$.

c. $\neg\neg(\phi \wedge \psi) \Rightarrow (\neg\neg\phi \wedge \neg\neg\psi)$.

d. $\neg(\phi \vee \psi) \Rightarrow (\neg\phi \wedge \neg\psi)$ *and* $(\neg\phi \wedge \neg\psi) \Rightarrow \neg(\phi \vee \psi)$.

3.4 Natural Deduction versus Deduction *à la* Gentzen

In this section we prove that both natural deduction and deduction *à la* Gentzen have the same expressive power, that means that we can prove exactly the same set of theorems using natural deduction or using deduction *à la* Gentzen. Initially, we prove that the property holds restricted to the intuitionistic logic. Then, we prove that it holds for the logic of predicates.

The main result is stated as

$$\vdash_G \Gamma \Rightarrow \varphi \quad \text{if and only if} \quad \Gamma \vdash_N \varphi$$

For proving this result, we will use an informal style of discussion which requires a deal of additional effort of the reader in order to interpret a few points that would not be presented in detail. Among others points, notice for instance that the antecedent "Γ" of the sequent $\Gamma \Rightarrow \varphi$ is in fact a *multiset* of formulas, while "Γ" as premise of $\Gamma \vdash_N \varphi$ should be interpreted as a *finite subset* of assumptions built from Γ that can be used in a natural derivation of φ.

Notice also, that in the classical sequent calculus one can build derivations for sequents of the form $\Gamma \Rightarrow \Delta$, and in natural deduction only derivations of a formula, say δ, are allowed, that is derivations of the form $\Gamma' \vdash_N \delta$. Then for the classical logic it would be necessary to establish a correspondence between derivability of arbitrary sequents of the form $\Gamma \Rightarrow \Delta$ and derivability of "equivalent" sequents with exactly one formula in the succedent of the form $\Gamma' \Rightarrow \delta$.

3.4.1 Equivalence between ND and Gentzen's SC - the intuitionistic case

The equivalence for the case of the intuitionistic logic is established in the next theorem.

Theorem 13 (ND versus SC for the intuitionistic logic)**.** *The equivalence below holds for the intuitionistic sequent calculus and the intuitionistic natural deduction:*

$$\vdash_G \Gamma \Rightarrow \varphi \textit{ if and only if } \Gamma \vdash_N \varphi$$

Proof. According to previous observations, it is possible to consider the calculus *à la* Gentzen without weakening rules. We will prove that the intuitionistic Gentzen's sequent calculus, including the cut rule, is equivalent to intuitionistic natural deduction. The proof is by induction on the structure of derivations.

Initially, we prove necessity, that is $\vdash_G \Gamma \Rightarrow \varphi$ implies $\Gamma \vdash_N \varphi$. This is done by induction on derivations in the intuitionistic Gentzen's sequent calculus, analyzing different cases according to the last rule applied in a derivation.

IB. The simplest derivations *à la* Gentzen are given by applications of rules (Ax) and ($L_\perp$):

$\Gamma, \varphi \Rightarrow \varphi$ (Ax) $\qquad \Gamma, \perp \Rightarrow \varphi$ ($L_\perp$)

In natural deduction, these proofs correspond respectively to derivations:

$$[\varphi]^u \text{ (Ax)} \qquad\qquad \frac{[\perp]^u}{\varphi}\ (\perp_e)$$

Notice that this means that $\Gamma, \varphi \vdash_N \varphi$ and $\Gamma, \perp \vdash_N \varphi$, since the assumption of the former derivation φ belongs to $\Gamma \cup \{\varphi\}$ and the assumption of the latter derivation $\perp$ belongs to $\Gamma \cup \{\perp\}$.

IS. We will consider derivations in the Gentzen calculus analyzing cases according to the last rule applied in the derivation. Right rules correspond to introduction rules, and left rules will need a more elaborated analysis. First, observe that in the intuitionistic case the sole contraction rule to be considered is (LC):

$$\dfrac{\overset{\nabla}{\psi, \psi, \Gamma \Rightarrow \varphi}}{\psi, \Gamma \Rightarrow \varphi}\ (\mathrm{LC})$$

And, whenever we have a derivation finishing in an application of this rule, by induction hypothesis, there is a natural derivation of its premise $\{\psi\} \cup \{\psi\} \cup \Gamma \vdash_N \varphi$, which corresponds to $\{\psi\} \cup \Gamma \vdash_N \varphi$ because the premises in natural deduction are sets.

Case ($\mathrm{L}_\wedge$). Suppose one has a derivation of the form

$$\dfrac{\overset{\nabla}{\psi, \Gamma \Rightarrow \varphi}}{\psi \wedge \delta, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\wedge)$$

By induction hypothesis, one has a derivation for $\Gamma, \psi \vdash_N \varphi$, say ∇', whose assumptions are ψ and a finite subset Γ' of Γ. Thus a natural derivation is obtained as below, by replacing each occurrence of the assumption $[\psi]$ in ∇' by an application of rule $(\wedge_e)$.

$$\begin{array}{c} \dfrac{[\psi \wedge \delta]^u}{\psi}\ (\wedge_e) \\ \nabla' \\ \varphi \end{array}$$

By brevity, in the previous derivation assumptions in Γ' were dropped, as will be done in all other derivations in this proof.

Case ($\mathrm{R}_\wedge$). Suppose $\varphi = \delta \wedge \psi$ and one has a derivation of the form

$$\dfrac{\overset{\nabla_1}{\Gamma \Rightarrow \delta} \qquad \overset{\nabla_2}{\Gamma \Rightarrow \psi}}{\Gamma \Rightarrow \delta \wedge \psi}\ (\mathrm{R}_\wedge)$$

By induction hypothesis, one has derivations for $\Gamma \vdash_N \delta$ and $\Gamma \vdash_N \psi$, say ∇'_1 and ∇'_2. Thus, a natural derivation is built from these derivations applying the rule $(\wedge_i)$ as below.

$$\dfrac{\begin{array}{c}\nabla_1'\\ \delta\end{array} \qquad \begin{array}{c}\nabla_2'\\ \psi\end{array}}{\delta \wedge \psi}\ (\wedge_i)$$

Case (L$_\vee$). Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla_1\\ \delta, \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\delta \vee \psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\vee)$$

By induction hypothesis, one has derivations ∇_1' and ∇_2' for $\delta, \Gamma \vdash_N \varphi$ and $\psi, \Gamma \vdash_N \varphi$. Thus, a natural derivation, that assumes $\delta \vee \psi$, is obtained from these derivations applying the rule $(\vee_e)$ as below.

$$\dfrac{[\delta \vee \psi]^u \qquad \begin{array}{c}[\delta]^v\\ \nabla_1'\\ \varphi\end{array} \qquad \begin{array}{c}[\psi]^w\\ \nabla_2'\\ \varphi\end{array}}{\varphi}\ (\vee_e)\ v, w$$

Case (R$_\vee$). Suppose $\varphi = \delta \vee \psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \Gamma \Rightarrow \delta\end{array}}{\Gamma \Rightarrow \delta \vee \psi}\ (\mathrm{R}_\vee)$$

By induction hypotheses there exists a natural derivation ∇' for $\Gamma \vdash_N \delta$. Applying at the end of this derivation rule $(\vee_i)$, one obtains a natural derivation for $\Gamma \vdash_N \delta \vee \psi$.

Case (L$_\rightarrow$). Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla_1\\ \Gamma \Rightarrow \delta\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\delta \rightarrow \psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\rightarrow)$$

By induction hypothesis there exist natural derivations ∇'_1 and ∇'_2 for $\Gamma \vdash_N \delta$ and $\psi, \Gamma \vdash_N \varphi$. A natural derivation for $\Gamma \vdash_N \varphi$ is obtained from these derivations, by replacing each assumption $[\psi]^u$ in ∇'_2 by a derivation of ψ finishing in an application of rule $(\rightarrow_e)$ with premises $[\delta \rightarrow \psi]^v$ and δ. The former as a new assumption and the latter is derived as in ∇'_1.

$$\dfrac{[\delta \rightarrow \psi]^v \qquad \begin{array}{c}\nabla'_1\\ \delta\end{array}}{\begin{array}{c}\psi\\ \nabla'_2\\ \varphi\end{array}} \ (\rightarrow_e)$$

Case $(\mathrm{R}_\rightarrow)$. Suppose $\varphi = \delta \rightarrow \psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \delta, \Gamma \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \delta \rightarrow \psi} \ (\mathrm{R}_\rightarrow)$$

By induction hypothesis, there exists a natural derivation ∇' for $\delta, \Gamma \vdash_N \psi$. The natural derivation for $\Gamma \vdash_N \delta \rightarrow \psi$ is obtained by applying at the end of this proof rule $(\rightarrow_i)$ discharging assumptions $[\delta]^u$ as depicted below.

$$\dfrac{\begin{array}{c}[\delta]^u\\ \nabla'\\ \psi\end{array}}{\delta \rightarrow \psi} \ (\rightarrow_i)\ u$$

Case $(\mathrm{L}_\forall)$. Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \psi[x/y], \Gamma \Rightarrow \varphi\end{array}}{\forall_x \psi, \Gamma \Rightarrow \varphi} \ (\mathrm{L}_\forall)$$

Then by induction hypothesis there exists a natural derivation for $\psi[x/y], \Gamma \vdash_N \varphi$, say ∇'.

A natural derivation for $\forall_x\psi, \Gamma \vdash_N \varphi$ is obtained by replacing all assumptions of $[\psi[x/y]]^u$ in ∇' by a deduction of $\psi[x/y]$ with assumption $[\forall_x\psi]^v$ applying rule $(\forall_e)$.

Case ($R_\forall$). Suppose $\varphi = \forall_x\psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \Gamma \Rightarrow \psi[x/y]\end{array}}{\Gamma \Rightarrow \forall_x\psi}\ (R_\forall)$$

where $y \notin \mathtt{fv}(\Gamma)$. Then by induction hypothesis there exists a natural derivation ∇' for $\Gamma \vdash_N \psi[x/y]$. Thus a simple application at the end of ∇' of rule $(\forall_i)$, that is possible since y does not appears in the open assumptions, will complete the desired natural derivation.

Case ($L_\exists$). Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \psi[x/y], \Gamma \Rightarrow \varphi\end{array}}{\exists_x\psi, \Gamma \Rightarrow \varphi}\ (L_\exists)$$

where $y \notin \mathtt{fv}(\Gamma, \varphi)$. By induction hypothesis there exists a natural derivation ∇' for $\psi[x/y], \Gamma \vdash_N \varphi$. The desired derivation is built by an application of rule $(\exists_e)$ using as premises the assumption $[\exists_x\psi]^v$ and the conclusion of ∇'. In this application assumptions of $[\psi[x/y]]^u$ in ∇' are discharged as depicted below. Notice that the application of rule $(\exists_e)$ is possible since $y \notin \mathtt{fv}(\Gamma, \varphi)$, which implies it does not will appear in open assumptions in ∇'.

$$\dfrac{[\exists_x\psi]^v \qquad \begin{array}{c}[\psi[x/y]]^u\\ \nabla'\\ \varphi\end{array}}{\varphi}\ (\exists_e)\ u$$

Case ($R_\exists$). Suppose $\varphi = \exists_x\psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \Gamma \Rightarrow \psi[x/t]\end{array}}{\Gamma \Rightarrow \exists_x\psi}\ (R_\exists)$$

A natural derivation for $\Gamma \vdash_N \exists_x \psi$ is built by induction hypothesis which gives a natural derivation ∇' for $\Gamma \vdash_N \psi[x/t]$ and application of rule $(\exists_i)$ to the conclusion of ∇'.

Case (cut). Suppose one has a derivation finishing in an application of rule (Cut) as below

$$\dfrac{\begin{array}{c}\nabla_1\\ \Gamma \Rightarrow \psi\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\Gamma \Rightarrow \varphi}\ (\text{Cut})$$

By induction hypothesis there are natural derivations ∇'_1 and ∇'_2 for $\Gamma \vdash_N \psi$ and $\psi, \Gamma \vdash_N \varphi$. To obtain the desired natural derivation, all assumptions $[\psi]^u$ in ∇'_2 are replaced by derivations of ψ using ∇'_1:

$$\begin{array}{c}\nabla'_1\\ \psi\\ \nabla'_2\\ \varphi\end{array}$$

Now we prove sufficiency, that is $\vdash_G \Gamma \Rightarrow \varphi$ whenever $\Gamma \vdash_N \varphi$. The proof is by induction on the structure of natural derivations analyzing the last applied rule.

IB. Proofs consisting of a sole node $[\varphi]^u$ correspond to applications of (Ax): $\Gamma \Rightarrow \varphi$, where $\varphi \in \Gamma$.

IS. All derivations finishing in introduction rules are straightforwardly related with derivations *à la* Gentzen finishing in the corresponding right rule as in the proof of necessity. Only one example is given: $(\rightarrow_i)$. The other cases are left as an exercise for the reader.

Suppose $\varphi = \delta \rightarrow \psi$ and one has a derivation finishing in an application of $(\rightarrow_i)$ discharging assumptions of δ and using assumptions in Γ:

$$\dfrac{\begin{array}{c}[\delta]^u\\ \nabla\\ \psi\end{array}}{\delta \rightarrow \psi}\ (\rightarrow_i)\ u$$

By induction hypothesis there exists a derivation *à la* Gentzen ∇' for the sequent $\delta, \Gamma \Rightarrow \psi$.

Thus, the desired derivation is built by a simple application of rule $(\mathrm{R}_\to)$:

$$\dfrac{\begin{array}{c}\nabla' \\ \delta, \Gamma \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \delta \to \psi}\ (\mathrm{R}_\to)$$

Derivations finishing in elimination rules will require application of the rule (Cut). A few interesting cases are given. All the other cases remain as an exercise for the reader.

Case $(\vee_e)$. Suppose one has a natural derivation for $\Gamma \vdash_N \varphi$ finishing in an application of rule $(\vee_e)$ as below.

$$\dfrac{\begin{array}{c}\nabla \\ \delta \vee \psi\end{array} \qquad \begin{array}{c}[\delta]^v \\ \nabla_1 \\ \varphi\end{array} \qquad \begin{array}{c}[\psi]^w \\ \nabla_2 \\ \varphi\end{array}}{\varphi}\ (\vee_e)\ v, w$$

By induction hypothesis, there are derivations *à la* Gentzen ∇', ∇'_1 and ∇'_2 respectively, for the sequents $\Gamma \Rightarrow \delta \vee \psi$, $\delta, \Gamma \Rightarrow \varphi$ and $\psi, \Gamma \Rightarrow \varphi$. Thus, using these derivations, a derivation for $\Gamma \Rightarrow \varphi$ is built as below.

$$\dfrac{\begin{array}{c}\nabla' \\ \Gamma \Rightarrow \delta \vee \psi\end{array} \qquad \dfrac{\begin{array}{c}\nabla'_1 \\ \delta, \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\nabla'_2 \\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\delta \vee \psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\vee)}{\Gamma \Rightarrow \varphi}\ (\mathrm{Cut})$$

Case $(\to_e)$. Suppose one has a natural derivation for $\Gamma \vdash_N \varphi$ that finishes in an application of $(\to_e)$ as below.

$$\dfrac{\begin{array}{c}\nabla_1 \\ \delta\end{array} \qquad \begin{array}{c}\nabla_2 \\ \delta \to \varphi\end{array}}{\varphi}\ (\to_e)$$

By induction hypothesis, there are derivations *à la* Gentzen ∇'_1 and ∇'_2 for the sequents

$\Gamma \Rightarrow \delta$ and $\Gamma \Rightarrow \delta \to \varphi$, respectively. The desired derivation is built, using these derivations, as below.

$$\dfrac{\begin{array}{c}\nabla'_2\\ \Gamma\Rightarrow\delta\to\varphi\end{array} \qquad \dfrac{\begin{array}{c}\nabla'_1\\ \Gamma\Rightarrow\delta\end{array}\qquad \varphi,\Gamma\Rightarrow\varphi\ (\mathrm{Ax})}{\delta\to\varphi,\Gamma\Rightarrow\varphi}\,(\mathrm{L}_\to)}{\Gamma\Rightarrow\varphi}\,(\mathrm{Cut})$$

Case $(\exists_e)$. Suppose one has a natural derivation for $\Gamma \vdash_N \varphi$ finishing in an application of the rule $(\exists_e)$ as below.

$$\dfrac{\begin{array}{c}\nabla_1\\ [\exists_x\psi]^v\end{array} \qquad \begin{array}{c}[\psi[x/y]]^u\\ \nabla_2\\ \varphi\end{array}}{\varphi}\,(\exists_e)\ u$$

By induction hypothesis, there are derivations *à la* Gentzen ∇'_1 and ∇'_2 for the sequents $\Gamma \Rightarrow \exists_x\psi$ and $\psi[x/y], \Gamma \Rightarrow \varphi$, respectively. The derivation is built as below. Notice that $y \notin \mathtt{fv}(\Gamma, \varphi)$, which allows the application of the rule $(\mathrm{L}_\exists)$.

$$\dfrac{\begin{array}{c}\nabla'_1\\ \Gamma\Rightarrow\exists_x\psi\end{array} \qquad \dfrac{\begin{array}{c}\nabla'_2\\ \psi[x/y],\Gamma\Rightarrow\varphi\end{array}}{\exists_x\psi,\Gamma\Rightarrow\varphi}\,(\mathrm{L}_\exists)}{\Gamma\Rightarrow\varphi}\,(\mathrm{Cut})$$

□

Exercise 46. *Prove all remaining cases in the proof of sufficiency of Theorem 13.*

3.4.2 Equivalence of ND and Gentzen's SC - the classical case

Before proving equivalence of natural deduction and deduction *à la* Gentzen for predicate logic, a few additional definitions and properties are necessary. First of all, we define a notion that makes it possible to transform any sequent in an equivalent one but with only one formula in its succedent.

By $\overline{\Gamma}$ we generically denote any sequence of formulas built from the formulas in the sequence Γ, replacing each formula in Γ by either its negation or, when the head symbol of the formula is the negation symbol, eliminating it from the formula. For instance, let $\Delta = \delta_1, \neg\delta_2, \neg\delta_3, \delta_4$, then $\overline{\Delta}$ might represent sequences as $\neg\delta_1, \neg\neg\delta_2, \delta_3, \neg\delta_4$; $\neg\delta_1, \delta_2, \delta_3, \neg\delta_4$, etc. This transformation is not only relevant for our purposes in this chapter, but also in computational frameworks, as we will see in the next chapter, in order to get rid automatically of negative formulas in sequents that appear in a derivation.

Definition 30 (*c*-equivalent sequents). *We will say that sequents $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \neg\varphi$ as well as $\Gamma \Rightarrow \Delta, \varphi$ and $\neg\varphi, \Gamma \Rightarrow \Delta$ are c-equivalent in one-step. The equivalence closure of this relation is called the c-equivalence relation on sequents and is denoted as $\equiv_{ce}$.*

According to the previous notational convention, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ and $\Gamma, \overline{\Delta'} \Rightarrow \Delta, \overline{\Gamma'}$ are *c*-equivalent; that is,

$$\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad \equiv_{ce} \quad \Gamma, \overline{\Delta'} \Rightarrow \Delta, \overline{\Gamma'}$$

Lemma 8 (One-step *c*-equivalence). *The following properties hold in the sequent calculus à la Gentzen for the classical logic:*

i) There exists a derivation for $\vdash_G \varphi, \Gamma \Rightarrow \Delta$, if and only if there exists a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \neg\varphi$.

ii) There is a derivation for $\vdash_G \neg\varphi, \Gamma \Rightarrow \Delta$, if and only if there is a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \varphi$.

Proof. We consider the derivations below.

i) **Necessity**: Let ∇ be a derivation for $\vdash_G \varphi, \Gamma \Rightarrow \Delta$. Then the desired derivation is built as follows:

$$\dfrac{\dfrac{\begin{array}{c}\nabla\\ \varphi, \Gamma \Rightarrow \Delta\end{array}}{\varphi, \Gamma \Rightarrow \Delta, \bot}\ (\mathrm{RW})}{\Gamma \Rightarrow \Delta, \neg\varphi}\ (\mathrm{R}_{\rightarrow})$$

Sufficiency: Let ∇ be a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \neg\varphi$. Then the desired derivation is built as follows:

$$\dfrac{(\text{LW})\ \dfrac{\begin{array}{c}\nabla\\ \Gamma \Rightarrow \Delta, \neg\varphi\end{array}}{\varphi, \Gamma \Rightarrow \Delta, \neg\varphi} \qquad \dfrac{(\text{Ax})\ \varphi, \Gamma \Rightarrow \Delta, \varphi \qquad \bot, \varphi, \Gamma \Rightarrow \Delta\ (\text{L}_\bot)}{\neg\varphi, \varphi, \Gamma \Rightarrow \Delta}\ (\text{L}_\to)}{\varphi, \Gamma \Rightarrow \Delta}\ (\text{Cut})$$

Observe that in both cases, when Δ is the empty sequence we have an intuitionistic proof.

ii) **Necessity**: Let ∇ be a derivation for $\vdash_G \neg\varphi, \Gamma \Rightarrow \Delta$. Then the desired derivation is built as follows:

$$\dfrac{\begin{array}{c}\nabla'\\ \Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi \to \varphi\end{array} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\nabla\\ \neg\varphi, \Gamma \Rightarrow \Delta\end{array}}{\neg\varphi, \Gamma \Rightarrow \Delta, \varphi, \bot}\ (\text{RW})}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi}\ (\text{R}_\to) \qquad \varphi, \Gamma \Rightarrow \Delta, \varphi\ (\text{Ax})}{\neg\neg\varphi \to \varphi, \Gamma \Rightarrow \Delta, \varphi}\ (\text{L}_\to)}{\Gamma \Rightarrow \Delta, \varphi}\ (\text{Cut})$$

where ∇' is the derivation below:

$$\dfrac{\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \Delta, \varphi, \varphi, \bot\ \ (\text{Ax})}{\Gamma \Rightarrow \Delta, \varphi, \varphi, \neg\varphi}\ (\text{R}_\to) \qquad \bot, \Gamma \Rightarrow \Delta, \varphi, \varphi\ (\text{L}_\bot)}{\neg\neg\varphi, \Gamma \Rightarrow \Delta, \varphi, \varphi}\ (\text{L}_\to)}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi \to \varphi}\ (\text{R}_\to)$$

Observe that this case is strictly classic because the left premise of (Cut), that is the derivation ∇', is essentially a proof of the sequent $\Rightarrow \neg\neg\varphi \to \varphi$ (Also, see examples 18 and 22).

Sufficiency: let ∇ be a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \varphi$. Then the desired derivation is built as follows:

$$\dfrac{\overset{\nabla}{\Gamma \Rightarrow \Delta, \varphi} \qquad \bot, \Gamma \Rightarrow \Delta \ (\mathrm{L}_\bot)}{\neg\varphi, \Gamma \Rightarrow \Delta} \ (\mathrm{L}_\to)$$

Observe that in this case, when Δ is the empty sequence we have an intuitionistic proof.

□

Corollary 3 (One-step c-equivalence in the intuitionistic calculus)**.** *The following properties hold in the* **intuitionistic** *calculus à la Gentzen:*

i) There is a derivation for $\vdash_G \varphi, \Gamma \Rightarrow$*, if and only if there is a derivation for* $\vdash_G \Gamma \Rightarrow \neg\varphi$.

ii) Assuming that $\Rightarrow \neg\neg\varphi \to \varphi$*, the existence of a derivation for* $\vdash_G \neg\varphi, \Gamma \Rightarrow$*, implies the existence of a derivation for* $\vdash_G \Gamma \Rightarrow \varphi$.

iii) There exist a derivation for $\vdash_G \neg\varphi, \Gamma \Rightarrow$*, whenever there is a derivation for* $\vdash_G \Gamma \Rightarrow \varphi$.

Proof. The proof is obtained from the proof of Lemma 8, according to the observations given in that proof. In particular for the item ii), the proof of sufficiency of the lemma is easily modified as below.

$$
\dfrac{\dfrac{(\text{Assumption}) \quad \Rightarrow \neg\neg\varphi \to \varphi}{\Gamma \Rightarrow \neg\neg\varphi \to \varphi}\,(\text{RW}) \qquad \dfrac{\dfrac{\dfrac{\neg\varphi, \Gamma \Rightarrow}{\neg\varphi, \Gamma \Rightarrow \bot}\,(\text{RW})}{\Gamma \Rightarrow \neg\neg\varphi}\,(\text{R}_\to) \qquad \varphi, \Gamma \Rightarrow \varphi\ (\text{Ax})}{\neg\neg\varphi \to \varphi, \Gamma \Rightarrow \varphi}\,(\text{L}_\to)}{\Gamma \Rightarrow \varphi}\,(\text{Cut})
$$

□

Exercise 47. *Complete the proof of the Corollary 3.*

Lemma 9 (c-equivalence)**.** *Let* $\Gamma \Rightarrow \Delta$ *and* $\Gamma' \Rightarrow \Delta'$ *be c-equivalent sequents, that is* $\Gamma \Rightarrow \Delta \equiv_{ce} \Gamma' \Rightarrow \Delta'$. *Then the following holds in the* **classical** *Gentzen's sequent calculus:*

$$\vdash_G \Gamma \Rightarrow \Delta \textit{ if and only if } \vdash_G \Gamma' \Rightarrow \Delta'$$

Proof. (Sketch) Suppose, $\Gamma \Rightarrow \Delta$ equals $\Gamma^1, \Gamma^2 \Rightarrow \Delta^1, \Delta^2$ and $\Gamma' \Rightarrow \Delta'$ equals $\Gamma^1, \overline{\Delta^2} \Rightarrow \Delta^1, \overline{\Gamma^2}$. The proof is by induction on $n = |\Gamma^2, \Delta^2|$, that is the number of switched formulas (from the succedent to the antecedent and vice versa), that are necessary to obtain $\Gamma' \Rightarrow \Delta'$ from $\Gamma \Rightarrow \Delta$ by a number n of one-step c-equivalence transformations. Suppose $\Gamma^1, \Gamma^2_k, \overline{\Delta^2_k} \Rightarrow \Delta^1, \Delta^2_k, \overline{\Gamma^2_k}$, for $0 \leq k \leq n$, is the sequent after k one-step c-equivalence transformations, being $\Gamma^2_0 = \Gamma^2$, $\Delta^2_0 = \Delta^2$ (thus, being $\overline{\Delta^2_0}$ and $\overline{\Gamma^2_0}$ empty sequences) and Γ^2_n and Δ^2_n empty sequences (thus, being $\overline{\Gamma^2_n} = \overline{\Gamma^2}$ and $\overline{\Delta^2_n} = \overline{\Delta^2}$).

In the inductive step, for $k < n$, one assumes that there is a proof of the sequent: $\vdash_G \Gamma^1, \Gamma^2_k, \overline{\Delta^2_k} \Rightarrow \Delta^1, \Delta^2_k, \overline{\Gamma^2_k}$. Thus, applying an one-step c-equivalence transformation, by Lemma 8, one obtains a proof for $\vdash_G \Gamma^1, \Gamma^2_{k+1}, \overline{\Delta^2_{k+1}} \Rightarrow \Delta^1, \Delta^2_{k+1}, \overline{\Gamma^2_{k+1}}$. □

Exercise 48. *Complete all details of the proof of Lemma 9.*

In order to extend the c-equivalence Lemma from classical to intuitionistic logic, it is necessary to assume all necessary stability axioms (Cf. item 2 of Corollary 3).

Definition 31 (Intuitionistic derivability modulo stability axioms)**.** *A stability axiom is a sequent of the form* $\Rightarrow \forall_x(\neg\neg\varphi \rightarrow \varphi)$*. Intuitionistic derivability modulo stability axioms is defined as intuitionistic derivability assuming all possible stability axioms. Intuitionistic derivability à la Gentzen with stability axioms will be denoted as* $\vdash_{Gi+St}$*.*

Lemma 10 (Equivalence between classical and intuitionistic SC modulo stability axioms)**.** *For all sequents* $\Gamma \Rightarrow \delta$ *the following property holds:*

$$\vdash_G \Gamma \Rightarrow \delta \textit{ iff } \vdash_{Gi+St} \Gamma \Rightarrow \delta$$

Therefore, for any sequent $\Gamma' \Rightarrow \Delta'$ *c-equivalent to* $\Gamma \Rightarrow \delta$*,* $\vdash_G \Gamma' \Rightarrow \Delta'$ *iff* $\vdash_{Gi+St} \Gamma \Rightarrow \delta$*.*

Proof. (Sketch) To prove that $\vdash_{Gi+St} \Gamma \Rightarrow \delta$ implies $\vdash_G \Gamma \Rightarrow \delta$, suppose that ∇ is a derivation for $\vdash_{Gi+St} \Gamma \Rightarrow \delta$. The derivation ∇ is transformed into a classical derivation in the following manner: for any stability axiom assumption, that is a sequent of the form $\Rightarrow \forall_x(\neg\neg\varphi \rightarrow \varphi)$ that appears as a leaf in the derivation ∇, replace the assumption by a classical proof for $\vdash_G \ \Rightarrow \forall_x(\neg\neg\varphi \rightarrow \varphi)$. In this way, after all stability axiom assumptions are replaced by classical derivations, one obtains a classical derivation, say ∇', for $\vdash_G \Gamma \Rightarrow \delta$. Additionally, by Lemma 9, $\vdash_G \Gamma \Rightarrow \delta$ if and only if there exists a classical derivation for $\vdash_G \Gamma' \Rightarrow \Delta'$.

To prove that $\vdash_{Gi+St} \Gamma \Rightarrow \delta$ whenever $\vdash_G \Gamma \Rightarrow \delta$, one applies induction on the structure of the classical derivation. Most rules require a direct analysis, for instance the inductive step for rule $(\mathrm{R}_\rightarrow)$ is given below.

Case $(\mathrm{R}_\rightarrow)$. The derivation is of the form given below.

$$\dfrac{\overset{\nabla}{\Gamma, \varphi \Rightarrow \psi}}{\Gamma \Rightarrow' \varphi \rightarrow \psi}\ (\mathrm{R}_\rightarrow)$$

By induction hypothesis there exists a derivation ∇' for $\vdash_{Gi+St} \Gamma, \varphi \Rightarrow \psi$, Thus, the desired derivation is obtained simply by an additional application of rule $(\mathrm{R}_\rightarrow)$ to the conclusion of the intuitionistic derivation ∇'.

The interesting case happens for rule ($L_\to$) since this rule requires two formulas in the succedent of one of the premises. The analysis of the inductive step for rule ($L_\to$) is given below.

Case ($L_\to$). The last step of the proof is of the form below, where $\Gamma = \Gamma'', \varphi \to \psi$.

$$\dfrac{\begin{matrix}\nabla_1 \\ \Gamma'' \Rightarrow \delta, \varphi\end{matrix} \qquad \begin{matrix}\nabla_2 \\ \psi, \Gamma'' \Rightarrow \delta\end{matrix}}{\Gamma'', \varphi \to \psi \Rightarrow \delta}\ (L_\to)$$

By induction hypothesis there exist derivations, say ∇_1' and ∇_2', for $\vdash_{Gi+St} \Gamma'', \neg\delta \Rightarrow \varphi$ and $\vdash_{Gi+St} \psi, \Gamma, \neg\delta \Rightarrow$. Notice that the argumentation is not as straightforwardly as it appears, since it is necessary to build first classical derivations for $\vdash_G \Gamma'', \neg\delta \Rightarrow \varphi$ and $\vdash_G \psi, \Gamma'', \neg\delta \Rightarrow$ using (Lemma 8 and) Corollary 3.

Thus, a derivation for $\vdash_{Gi+St} \Gamma'', \varphi \to \psi, \neg\delta \Rightarrow$ is obtained as below.

$$\dfrac{\begin{matrix}\nabla_1' \\ \Gamma'', \neg\delta \Rightarrow \varphi\end{matrix} \qquad \begin{matrix}\nabla_2' \\ \psi, \Gamma'', \neg\delta \Rightarrow\end{matrix}}{\Gamma'', \varphi \to \psi, \neg\delta \Rightarrow}\ (L_\to)$$

By a final application of Corollary 3 there exists a derivation for $\vdash_{Gi+St} \Gamma, \varphi \to \psi \Rightarrow \delta$. □

Exercise 49. *Prove the remaining cases of the proof of Lemma 10.*

Theorem 14 (Natural versus deduction *à la* Gentzen for the classical logic)**.** *One has that for the classical Gentzen and natural calculus*

$$\vdash_G \Gamma \Rightarrow \varphi \textit{ if and only if } \Gamma \vdash_N \varphi$$

Proof. (Sketch) By previous Lemma, $\vdash_G \Gamma \Rightarrow \varphi$ if and only if $\vdash_{Gi+St} \Gamma \Rightarrow \varphi$. Thus, we only require to prove that $\vdash_{Gi+St} \Gamma \Rightarrow \varphi$ if and only if $\Gamma \vdash_N \varphi$.

On the one side, an intuitionistic sequent calculus derivation modulo stability axioms for

$\Gamma \Rightarrow \varphi$ will include some assumptions of the form $\Rightarrow \forall_x(\neg\neg\varphi_i \to \varphi_i)$, for formulas φ_i, with $i \leq k$ for some k in $\mathbb{N}$. Thus, by Theorem 13 there exists an intuitionistic proof in natural deduction using these stability axioms as assumptions. This intuitionistic natural derivation is converted into a classical derivation by including classical natural derivations for these assumptions.

On the other side, suppose that $\Gamma \vdash_N \varphi$ and let us assume that ∇ is a natural derivation for $\Gamma \vdash_N \varphi$ that uses only the classical rule $(\neg\neg_e)$; that is ∇ has no application of other exclusively classical rules such as (PBC) or (LEM). The derivation ∇ is transformed into an intuitionistic derivation with assumptions of stability axioms by applying to any application of the rule $(\neg\neg_e)$ in ∇, the following transformation:

$$\dfrac{\begin{array}{c}\nabla' \\ \neg\neg\varphi\end{array}}{\varphi}\,(\neg\neg_e) \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\nabla' \\ \neg\neg\varphi\end{array} \qquad \dfrac{[\forall_x(\neg\neg\varphi \to \varphi)]^u}{\neg\neg\varphi \to \varphi}\,(\forall_e)}{\varphi}\,(\to_e)$$

In this manner, after replacing all applications of the rule $(\neg\neg_e)$, one obtains an intuitionistic natural derivation that has the original assumptions in Γ plus other assumptions that are stability axioms, say $\Gamma' = \forall_{x_1}(\neg\neg\varphi_1 \to \varphi_1), \ldots, \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k)$, for some k in $\mathbb{N}$. By Theorem 13 there exists an intuitionistic derivation a la Gentzen, say ∇'', for $\vdash_{Gi} \Gamma, \Gamma' \Rightarrow \varphi$. To conclude, note that one can get rid of all formulas in Γ' by using stability axioms of the form $\Rightarrow \forall_{x_i}(\neg\neg\varphi_i \to \varphi_i)$, for $i = 1, .., k$, and applications of the (Cut) rule as depicted below.

$$\dfrac{\Rightarrow \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k) \qquad \begin{array}{c} \dfrac{\Rightarrow \forall_{x_1}(\neg\neg\varphi_1 \to \varphi_1) \qquad \begin{array}{c}\nabla'' \\ \Gamma, \Gamma' \Rightarrow \varphi\end{array}}{\Gamma, \forall_{x_2}(\neg\neg\varphi_2 \to \varphi_2), \ldots, \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k) \Rightarrow \varphi}\,(\text{Cut}) \\ \vdots \quad k \text{ applications of (Cut)} \\ \Gamma, \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k) \Rightarrow \varphi \end{array}}{\Gamma \Rightarrow \varphi}\,(\text{Cut})$$

This gives the desired derivation for $\vdash_{Gi+St} \Gamma \Rightarrow \varphi$. □

Exercise 50. *Prove all details of Theorem 14.*