theses is greater than or equal to the number of closed parentheses in s.

2. *Any proper prefix s of a well-formed propositional formula ϕ might not be a well-formed propositional formula. By "proper" we understand that s can not be equal to ϕ.*

1.4 Natural Deductions and Proofs in the Propositional Logic

In this section, we will show that the goal of natural deduction is to deduce new information from facts that we already know, that we call *hypotheses* or *premises*. From now on, we will ignore external parentheses of formulas, whenever they do not introduce ambiguities. Suppose a set of formulas $S = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ (for some $n > 0$) is given, and we want to know if the formula ψ can be obtained from S. We start with a simple reasoning, with $n = 2$: Suppose that the formulas φ_1 and φ_2 hold. In this case, we can conclude that the formula $\varphi_1 \wedge \varphi_2$ also holds (according to the usual meaning of the conjunction). This kind of reasoning is "natural" and can be represented by a nice mathematical notation as follows:

$$\frac{\phi_1 \qquad \phi_2}{\phi_1 \wedge \phi_2}$$

The formulas above the line are the *premises*, while the one below the line corresponds to the *conclusion*, i.e., the **new** information inferred from the premises.

Similarly, if we know that $\varphi_1 \wedge \varphi_2$ is true then so is φ_1, and also φ_2. This piece of reasoning can be represented by the following rules

$$\frac{\phi_1 \wedge \phi_2}{\phi_1} \qquad\qquad \frac{\phi_1 \wedge \phi_2}{\phi_2}$$

With these three simple rules we can already *prove* a basic property of the conjunction:

the commutativity, i.e. if $\varphi \wedge \psi$ then $\psi \wedge \varphi$. A proof is a tree whose leafs are premises and whose root is the conclusion. The internal nodes of the tree correspond to applications of the rules: any internal node is labelled by a formula that is the conclusion of the formulas labeling its ancestral nodes.

$$\dfrac{\dfrac{\phi \wedge \psi}{\psi} \qquad \dfrac{\phi \wedge \psi}{\phi}}{\psi \wedge \phi}$$

In the above tree, the hypothesis $\varphi \wedge \psi$ is used twice, and the conclusion is $\psi \wedge \varphi$. In other words, we have proved that $\varphi \wedge \psi \vdash \psi \wedge \varphi$. In general, we call an expression of the form $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ a *sequent*. The formulas before the symbol $\vdash$ are the premises, and the one after, is the conclusion.

The system of natural deduction is composed by a set of inference rules. The idea is that each connective has an *introduction* and an *elimination* rule. Let see how it works for each connective. As we have seen, for the conjunction, the introduction rule is given by:

$$\dfrac{\phi_1 \qquad \phi_2}{\phi_1 \wedge \phi_2}\ (\wedge_i)$$

and two elimination rules:

$$\dfrac{\phi_1 \wedge \phi_2}{\phi_1}\ (\wedge_{e_1}) \qquad\qquad \dfrac{\phi_1 \wedge \phi_2}{\phi_2}\ (\wedge_{e_2})$$

The last two rules of elimination for the conjunction might be abbreviated as the unique rule:

$$\frac{\phi_1 \wedge \phi_2}{\phi_i \ \ {\scriptstyle (i=1,2)}} \ (\wedge_e)$$

The rules for implication are very intuitive: consider the following sentence

"if it is raining then driving is dangerous"

So, what one might conclude if it is raining? That driving is dangerous, of course. This kind of reasoning can be represented by an inference rule known as *modus ponens* (or elimination of the implication):

$$\frac{\phi \qquad \phi \to \psi}{\psi} \ (\to_e)$$

In order to introduce the implication $\phi \to \psi$ one needs to assume the premise of the implication, ϕ, and prove its conclusion, ψ. The (temporary) assumption ϕ is *discharged* once one introduces the implication, as depicted below:

$$\frac{\begin{matrix}[\phi]^a \\ \vdots \\ \psi\end{matrix}}{\phi \to \psi} \ (\to_i) \ a$$

In this rule $[\phi]^a$ denotes the set of all leaves in the deduction of ψ where the formula ϕ was assumed. Thus, the label "a" is related with the set of all these assumptions in the derivation tree of ψ. And the application of the rule $(\to_i)$ uses this label "a" to denote that all these assumptions are *closed* or *discharged* after the conclusion $\phi \to \psi$ is derived.

The $(\to_i)$ rule can also be applied without discharging any assumption: if one knows ψ then $\phi \to \psi$ holds, for any ϕ. In this case application of the rule is labelled with $(\to_i)\ \emptyset$. The use of the *empty set* symbol as label is justified since a label "a", as explained before, is related with the set of all assumptions of ϕ in the derivation tree labelled with a. The intuition behind this reasoning can be explained by the following example: suppose that we known that "I cannot fall asleep", then both "I drink coffee implies that I cannot fall asleep" and "I don't drink coffee implies that I cannot fall asleep" hold. That is, using the previous notation, one obtains the following derivations, where r and p mean respectively, "I cannot fall asleep" and "I drink coffee":

$$\dfrac{r}{p \to r}\ (\to_i)\ \emptyset \qquad\qquad \dfrac{r}{\neg p \to r}\ (\to_i)\ \emptyset$$

Introduction of the implication without discharging premises can be also be derived from an application of the rule with discharge of assumption as below:

$$\dfrac{\dfrac{\dfrac{\psi \qquad [\phi]^a}{\psi \wedge \phi}\ (\wedge_i)}{\psi}\ (\wedge_e)}{\phi \to \psi}\ (\to_i)\ a$$

Application of rules with temporary assumptions can discharge either none or several occurrences of the assumed formula. For instance, consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^z \qquad \dfrac{[\phi \to \phi \to \psi]^x \qquad [\phi]^y}{\phi \to \psi}\,(\to_e)}{\psi}\,(\to_e)}{\phi \to \psi}\,(\to_i)\ z}{\phi \to \phi \to \psi}\,(\to_i)\ y}{(\phi \to \phi \to \psi) \to \phi \to \phi \to \psi}\,(\to_i)\ x$$

In the above example, the temporary assumption ϕ was partially discharged in the first application of the rule $(\to_i)$ since only the assumption of the formula ϕ with label z was discharged, but not the assumption with label y. A logical system that allows this kind of derivation is said to obey the *partial discharge convention*. The above derivation can be solved with a complete discharge of the temporary assumption ϕ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^y \qquad \dfrac{[\phi \to \phi \to \psi]^x \qquad [\phi]^y}{\phi \to \psi}\,(\to_e)}{\psi}\,(\to_e)}{\phi \to \psi}\,(\to_i)\ \emptyset}{\phi \to \phi \to \psi}\,(\to_i)\ y}{(\phi \to \phi \to \psi) \to \phi \to \phi \to \psi}\,(\to_i)\ x$$

A logical system that forbids a partial discharge of temporary assumptions is said to obey the *complete discharge convention*. A comparison between the last two proofs suggests that partial discharges can be replaced by one complete discharge followed by vacuous ones. This is correct and so these discharge conventions play "little role in standard accounts of natural deduction", but it is relevant in *type theory* for the correspondence between proofs and λ-terms because "different discharge-labels will correspond to different terms". For more details, see suggested readings and references on type theory (Chapter 6).

For the disjunction, the introduction rules are given by:

$$\frac{\phi_1}{\phi_1 \vee \phi_2}\ (\vee_{i_1}) \qquad\qquad \frac{\phi_2}{\phi_1 \vee \phi_2}\ (\vee_{i_2})$$

The first introduction rule means that, if ϕ_1 holds, or in other words, if one has a proof of ϕ_1, then $\phi_1 \vee \phi_2$ also holds, where ϕ_2 is any formula. The meaning of the rule $(\vee_{i_2})$ is similar. As for the elimination of conjunction rule $(\wedge_e)$, these two rules might be abbreviated as a unique one:

$$\frac{\phi_i \quad {\scriptstyle (i=1,2)}}{\phi_1 \vee \phi_2}\ (\vee_i)$$

As another example of simultaneous discharging of occurrences of an assumption, observe the derivation for $\vdash (\phi \to ((\phi\vee\psi)\wedge(\phi\vee\varphi)))$ in which, by application of the rule of introduction of implication $(\to_i)$, two occurrences of the assumption of ϕ are discharged.

$$\dfrac{\dfrac{(\vee_i)\ \dfrac{[\phi]^u}{(\phi \vee \psi)} \qquad \dfrac{[\phi]^u}{(\phi \vee \varphi)}\ (\vee_i)}{((\phi \vee \psi) \wedge (\phi \vee \varphi))}\ (\wedge_i)}{(\phi \to ((\phi \vee \psi) \wedge (\phi \vee \varphi)))}\ (\to_i)\ u$$

The elimination rule for the disjunction is more subtle because from the fact that $\phi_1 \vee \phi_2$ holds, one does not know if ϕ_1, ϕ_2 or both ϕ_1 and ϕ_2 hold. Nevertheless, if a formula χ can be proved from ϕ_1 and also from ϕ_2 then it can be derived from $\phi_1 \vee \phi_2$. This is the idea of the elimination rule for the disjunction that is presented below. In this rule, the notation

$[\phi_1]^a$ means that ϕ_1 is a temporary assumption, or a hypothesis. Note that the rule scheme $(\vee_e)$ is labelled with a, b which means that the temporary assumptions are discharged, i.e., the assumptions are closed after the rule is applied.

$$\dfrac{\phi_1 \vee \phi_2 \qquad \begin{matrix}[\phi_1]^a \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\phi_2]^b \\ \vdots \\ \chi\end{matrix}}{\chi}\ (\vee_e)\ a, b$$

As an example consider the following reasoning: You know that both coffee and tea have caffeine, so if you drink one or the other you will not be able to fall asleep. This reasoning can be seen as an instance of the disjunction elimination as follows: Let p be a proposition whose meaning is "I drink coffee", q means "I drink tea" and r means "I cannot fall asleep". One can prove r as follows:

$$\dfrac{p \vee q \qquad (\to_e)\ \dfrac{[p]^a \qquad p \to r}{r} \qquad \dfrac{[q]^b \qquad q \to r}{r}\ (\to_e)}{r}\ (\vee_e)\ a, b$$

The above tree has 5 leafs:

1. the hypothesis $p \vee q$
2. the temporary assumption p
3. the fact $p \to r$ whose meaning is "if I drink coffee then I will not sleep".
4. the temporary assumption q
5. The temporary assumption $q \to r$ whose meaning is "if I drink tea then I would not fall asleep".

We need to assume p and q as "temporary" assumptions because we want to show that r is true independently of which one holds. We know that at least one of these propositions holds since we have that $p \vee q$ holds. Once the rule of elimination of disjunction is applied these temporary assumptions are discharged.

Exercise 4. *Prove that $\phi \vee \psi \vdash \psi \vee \phi$, i.e., the disjunction is commutative.*

For the negation, the rules are as follows:

$$\dfrac{\begin{array}{c}[\phi]^a \\ \vdots \\ \bot\end{array}}{\neg\phi}\ (\neg_i)\ a \qquad\qquad \dfrac{\phi \qquad \neg\phi}{\bot}\ (\neg_e)$$

The introduction rule says that if one is able to prove $\bot$ (the absurd) from the assumption ϕ, then $\neg\phi$ holds. This rule discharges the assumption ϕ concluding $\neg\phi$. The elimination rule states that if one is able to prove both a formula ϕ and its negation $\neg\phi$ then one can conclude the absurd $\bot$.

Remark 1. *Neither the symbol of negation $\neg$ nor the symbol $\top$ are necessary. $\top$ can be encoded as $\bot \to \bot$ and negation of a formula ϕ as $\phi \to \bot$. From this encoding, one can notice that rule $(\neg_e)$ is not essential; namely, it corresponds to an application of rule $(\to_e)$:*

$$\dfrac{\phi \qquad \phi \to \bot}{\bot}\ (\to_e)$$

Similarly, one can notice that rule $(\neg_i)$ is neither essential because it corresponds to an application of rule $(\to_i)$:

$$\dfrac{\begin{matrix}[\phi]^a \\ \vdots \\ \bot\end{matrix}}{\phi \to \bot}\ (\to_i)\ a$$

The absurd has no introduction rule, but it has an elimination rule, which corresponds to the application of rule $(\neg_i)$ discharging an empty set of assumptions:

$$\dfrac{\bot}{\phi}\ (\bot_e)$$

The set of rules presented so far (summarized in Table 1.2) represents a fragment of the propositional calculus known as the *intuitionistic propositional calculus*, which is considered as the logical basis of the constructive mathematics. The set of formulas derived from these rules are known as the *intuitionistic propositional logic*. Only the essential rules are presented, omitting for instance rules for introduction of disjunction to the right and elimination of conjunction to the right, since both the logical operators $\wedge$ and $\vee$ were proved to be commutative. Also derived rules are omitted. In particular, the rule $(\bot_e)$ is also known as the *intuitionistic absurdity rule*. Eliminating $(\bot_e)$ one obtains the *minimal propositional calculus*. The formulas derived from these rules are known as the *minimal propositional logic*.

Shortly, one can say that the constructive mathematics is the mathematics without the *law of the excluded middle* $(\varphi \vee \neg\varphi)$, denoted by (LEM) for short. In this theory one replaces the phrase "there exists" by "we can construct", which is particularly interesting for Computer Science. The law of the excluded middle is also known as the *law of the excluded third* which means that no third option is allowed (*tertium non datur*).

Remark 2. *There exists a fragment of the intuitionistic propositional logic that is of great interest in Computer Science. This is known as the implicational fragment of the propositional logic, and it contains only the rules* $(\rightarrow_i)$ *and* $(\rightarrow_e)$*. The computational interest in this fragment is that it is directly related to type inference in the functional paradigm of programming. In this paradigm (untyped) programs can be seen as terms of the following language:*

$$t ::= x \mid (t\ t) \mid (\lambda_x.t)$$

where x *ranges over a set of term variables,* $(t\ u)$ *represents the application of the function* t *to the argument* u*, and* $(\lambda_x.t)$ *represents a function with parameter* x *and body* t*. The construction* $(\lambda_x.t)$ *is called an abstraction. Types are either atomic of functional and their syntax is given as:*

$$\tau ::= \tau \mid \tau \rightarrow \tau$$

The type of a variable is annotated as $x : \tau$ *and a context* Γ *is a finite set of type annotations for variables in which each variable has a unique type.*

The simple typing rules for the above language are as follows:

$$\frac{\Gamma \vdash t : A \rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash (t\ u) : B}\ (\textsc{App}) \qquad \frac{\Gamma \cup \{x : A\} \vdash t : B}{\Gamma \vdash (\lambda_x.t) : A \rightarrow B}\ (\textsc{Abs})$$

$$\frac{}{\Gamma \vdash x : A}\ (\textsc{Var}),\ x : A \in \Gamma$$

Notice that, if one erases the term information on the rule (App)*, one gets exactly the rule* $(\rightarrow_e)$*. Similarly, the type information of the rule* (Abs) *corresponds to the rule* $(\rightarrow_i)$*. The rule* (Var) *does not correspond to any rule in natural deduction, but to a single assumption* $[A]^x$*, that is a derivation of* $A \vdash A$*. As an example, suppose one wants to build a function that computes the sum of two natural numbers* x *and* y*. That* x *and* y *are naturals is expressed through the type annotations* $x : \mathbb{N}$ *and* $y : \mathbb{N}$*. Thus, supposing one has proved that the* *function* add *has functional type* $\mathbb{N} \to \mathbb{N} \to \mathbb{N}$ *under context* $\Gamma = \{x : \mathbb{N}, y : \mathbb{N}\}$*, one can derive that* (add x y) *has type* $\mathbb{N}$ *under the same context as follows:*

$$\dfrac{(\textsc{App})\ \dfrac{\Gamma \vdash \texttt{add} : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \qquad \Gamma \vdash x : \mathbb{N}\ (\textsc{Var})}{\Gamma \vdash (\texttt{add}\ x) : \mathbb{N} \to \mathbb{N}} \qquad (\textsc{Var})\ \Gamma \vdash y : \mathbb{N}}{\Gamma \vdash ((\texttt{add}\ x)\ y) : \mathbb{N}}\ (\textsc{App})$$

The abstraction of the function projection of the first argument of a pair of naturals is built in this language as $(\lambda_x.(\lambda_y.x))$ *and its type is derived as follows:*

$$\dfrac{(\textsc{Abs})\ \dfrac{(\textsc{Var})\ \Gamma \vdash x : \mathbb{N}}{\{x : \mathbb{N}\} \vdash (\lambda_y.x) : \mathbb{N} \to \mathbb{N}}}{\vdash ((\lambda_x.(\lambda_y.x)) : \mathbb{N} \to \mathbb{N} \to \mathbb{N}}\ (\textsc{Abs})$$

For a detailed presentation on this subject, see the suggested readings and references on type theory.

The exclusion of (LEM) in the intuitionistic logic means that $(\varphi \vee \neg\varphi)$ holds only if one can prove either φ or $\neg\varphi$, while in *classical logic*, it is taken as an axiom. The classical logic can be seen as an extension of the intuitionistic logic, and hence there are sequents that are provable in the former, but not in the latter. The standard example of propositional formula that is provable in classical logic, but cannot be proved in intuitionistic logic is Peirce's law: $((\varphi \to \psi) \to \varphi) \to \varphi$.

It is relevant to stress here that in the classical propositional calculus the rule $(\bot_e)$ can discharge a non empty set of negative assumptions. This is not the case in the propositional intuitionistic calculus in which this rule can only be applied without discharging assumptions. Thus, the rules for the *propositional classical calculus* include a new rule for *proving by contradiction*, for short (PBC), in which after deriving the absurd one can discharge negative assumptions. Essentially, replacing ($\bot_e$) by (PBC) one obtains the calculus of natural deduction for the classical propositional logic (see Table 1.3).

Table 1.2: Rules of natural deduction for intuitionistic propositional logic

introduction rules	elimination rules
$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge_i)$	$\dfrac{\varphi \wedge \psi}{\varphi}\ (\wedge_e)$
$\dfrac{\varphi}{\varphi \vee \psi}\ (\vee_i)$	$\begin{array}{ccc} & [\varphi]^u & [\psi]^v \\ & \vdots & \vdots \\ \varphi \vee \psi & \chi & \chi \\ \hline & \chi & \end{array}\ (\vee_e)\ u, v$
$\begin{array}{c} [\varphi]^u \\ \vdots \\ \psi \\ \hline \varphi \to \psi \end{array}\ (\to_i)\ u$	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}\ (\to_e)$
$\begin{array}{c} [\varphi]^u \\ \vdots \\ \bot \\ \hline \neg\varphi \end{array}\ (\neg_i)\ u$	$\dfrac{\varphi \quad \neg\varphi}{\bot}\ (\neg_e)$
	$\dfrac{\bot}{\varphi}\ (\bot_e)$

In general, in order to get classical logic, one can add to the set of rules of Table 1.2 one of the following rules, where the rules ($\neg\neg_e$) and (LEM) are called respectively the rule of elimination of the double negation and rule for the law of middle excluded.

$$\dfrac{\neg\neg\phi}{\phi}\ (\neg\neg_e) \qquad\qquad \dfrac{}{\phi \vee \neg\phi}\ (\text{LEM}) \qquad\qquad \begin{array}{c} [\neg\phi]^a \\ \vdots \\ \bot \\ \hline \phi \end{array}\ (\text{PBC})\ a$$

In fact, any two of these rules can be proved from the third one. Assuming $(\neg\neg_e)$ one can prove (LEM) and (PBC):

$$
\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^x \qquad \dfrac{\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^x \qquad \dfrac{[\phi]^u}{\phi \vee \neg\phi}\,(\vee_i)}{\bot}\,(\neg_e)}{\neg\phi}\,(\neg_i)\ u}{\phi \vee \neg\phi}\,(\vee_i)}{\dfrac{\bot}{\neg\neg(\phi \vee \neg\phi)}\,(\neg_i)\ x}\,(\neg_e)}{\phi \vee \neg\phi}\,(\neg\neg_e)
$$

$$
\dfrac{\dfrac{\begin{array}{c}[\neg\phi]^a \\ \vdots \\ \bot\end{array}}{\neg\neg\phi}\,(\neg_i)\ a}{\phi}\,(\neg\neg_e)
$$

One can also prove (LEM) and $(\neg\neg_e)$ from (PBC):

$$\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^a \qquad \dfrac{\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^a \qquad \dfrac{[\neg\phi]^b}{(\phi \vee \neg\phi)}\,(\vee_i)}{\bot}\,(\neg_e)}{\phi}\,(\text{PBC})\ b}{\phi \vee \neg\phi}\,(\vee_i)}{\bot}\,(\neg_e)}{\phi \vee \neg\phi}\,(\text{PBC})\ a$$

$$\dfrac{\dfrac{\neg\neg\phi \qquad [\neg\phi]^a}{\bot}\,(\neg_e)}{\phi}\,(\text{PBC})\ a$$

Finally, from (LEM) one can prove $(\neg\neg_e)$ and (PBC):

$$\dfrac{(\text{LEM})\,\dfrac{}{\phi \vee \neg\phi} \qquad \dfrac{\dfrac{[\neg\phi]^a \qquad \neg\neg\phi}{\bot}\,(\neg_e)}{\phi}\,(\bot_e) \qquad [\phi]^b}{\phi}\,(\vee_e)\ a, b$$

$$\dfrac{(\text{LEM})\,\dfrac{}{\phi \vee \neg\phi} \qquad [\phi]^a \qquad \dfrac{\begin{array}{c}[\neg\phi]^b\\ \vdots \\ \bot\end{array}}{\phi}\,(\bot_e)}{\phi}\,(\vee_e)\ a, b$$

Table 1.3 includes the set of natural deduction rules for the classical propositional logic where our preference was to add the rule (PBC). Note that the rule $(\bot_e)$ can be removed from Table 1.3 because it can be deduced directly from (PBC) by an empty discharge.

Exercise 5. *Prove that the rule $(\bot_e)$ is not essential, i.e., prove that this rule can be derived from the rules presented in Table 1.3.*

There are several proofs that are useful in many situations. These proofs are pieced together to build more elaborated pieces of reasoning. For this reason, these proofs will be added as *derived rules* in our natural deduction system. The first one, is for the introduction of the double negation: $\varphi \vdash \neg\neg\varphi$.

$$\dfrac{\dfrac{\varphi \qquad [\neg\varphi]^a}{\bot}\,(\neg_e)}{\neg\neg\varphi}\,(\neg_i)\ a$$

The corresponding derived rule is as follows:

$$\dfrac{\phi}{\neg\neg\phi}\,(\neg\neg_i)$$

Table 1.3: RULES OF NATURAL DEDUCTION FOR CLASSICAL PROPOSITIONAL LOGIC

introduction rules	elimination rules
$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge_i)$	$\dfrac{\varphi \wedge \psi}{\varphi}\ (\wedge_e)$
$\dfrac{\varphi}{\varphi \vee \psi}\ (\vee_i)$	$\dfrac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]^u \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi]^v \\ \vdots \\ \chi\end{array}}{\chi}\ (\vee_e)\ u, v$
$\dfrac{\begin{array}{c}[\varphi]^u \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\ (\to_i)\ u$	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}\ (\to_e)$
$\dfrac{\begin{array}{c}[\varphi]^u \\ \vdots \\ \bot\end{array}}{\neg\varphi}\ (\neg_i)\ u$	$\dfrac{\varphi \quad \neg\varphi}{\bot}\ (\neg_e)$
	$\dfrac{\begin{array}{c}[\neg\varphi]^u \\ \vdots \\ \bot\end{array}}{\varphi}\ (\text{PBC})\ u$

Once, a derivation is done, new rules can be included to the set of applicable ones.

Another rule of practical interest is *modus tollens*, that states that whenever one knows that $\phi \to \psi$ and $\neg\psi$, $\neg\phi$ holds. For instance if we know both that "if Aristotle was Indian then he was Asian" and that "he wasn't Asian", then we have that "Aristotle wasn't Indian". *Modus tollens*, that is $(\neg\psi), (\phi \to \psi) \vdash (\neg\phi)$, can be derived as follows.

$$
\cfrac{\cfrac{\cfrac{[\phi]^x \qquad (\phi \to \psi)}{\psi}\,(\to_e) \qquad (\neg\psi)}{\bot}\,(\neg_e)}{(\neg\phi)}\,(\neg_i)\ x
$$

Thus, a new derived rule for *modus tollens* can be added:

$$
\frac{(\neg\psi) \qquad (\phi \to \psi)}{(\neg\phi)}\,(\text{MT})
$$

Another useful derived rules are the *contrapositive* ones. In particular, proving an implication $(\phi \to \psi)$ by *contraposition* consists of proving $(\neg\psi \to \neg\phi)$ or vice versa. Thus, in order to use this reasoning mechanism, it is necessary to build derivations for $(\phi \to \psi) \vdash (\neg\psi \to \neg\phi)$ as well as for $(\neg\psi \to \neg\phi) \vdash (\phi \to \psi)$. A derivation for the former sequent is presented below.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\phi \to \psi \qquad [\phi]^y}{\psi}\,(\to_e) \qquad [\neg\psi]^x}{\bot}\,(\neg_e)}{\neg\phi}\,(\neg_i)\ y}{\neg\psi \to \neg\phi}\,(\to_i)\ x
$$

A derivation of the latter sequent is presented below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\psi \to \neg\phi \qquad [\neg\psi]^y}{\neg\phi}\,(\to_e) \qquad [\phi]^x}{\bot}\,(\neg_e)}{\psi}\,(\text{PBC})\ y}{\phi \to \psi}\,(\to_i)\ x$$

Thus, new derived rules for contraposition, for short (CP), can be given as:

$$\dfrac{\phi \to \psi}{\neg\psi \to \neg\phi}\,(\text{CP}_1) \qquad\qquad \dfrac{\neg\psi \to \neg\phi}{\phi \to \psi}\,(\text{CP}_2)$$

A few interesting rules that can be derived from the natural deduction calculus (as given in Table 1.3) are presented in Table 1.4.

Table 1.4: Derived rules of natural deduction for propositional logic

$\dfrac{}{\varphi \vee \neg\varphi}\,(\text{LEM})$	
$\dfrac{\neg\neg\varphi}{\varphi}\,(\neg\neg_e)$	$\dfrac{\varphi}{\neg\neg\varphi}\,(\neg\neg_i)$
$\dfrac{\psi \to \varphi \qquad \neg\varphi}{\neg\psi}\,(\text{MT})$	$\dfrac{\bot}{\varphi}\,(\bot_e)$
$\dfrac{\varphi \to \psi}{\neg\psi \to \neg\varphi}\,(\text{CP}_1)$	$\dfrac{\neg\varphi \to \neg\psi}{\psi \to \varphi}\,(\text{CP}_2)$

Definition 8 (Formulas provable equivalent)**.** *Let ϕ and ψ well-formed propositional formulas. Whenever, one has that $\phi \vdash \psi$ and also that $\psi \vdash \phi$, it is said that ϕ and ψ are provable equivalent. This is denoted as $\phi \dashv\vdash \psi$.*

Notice that $\phi \to \psi \dashv\vdash \neg\psi \to \neg\phi$.

Exercise 6. *Build derivations for both versions of contraposition below.*

a. $\neg\psi \to \phi \dashv\vdash \neg\phi \to \psi$ *and*

b. $\psi \to \neg\phi \dashv\vdash \phi \to \neg\psi$.

In the sequel, several examples are presented.

Example 5 (Associativity of conjunction and disjunction)**.** *Derivations of the associativity of conjunction and disjunction are presented.*

- *First, the associativity of conjunction is proved; that is,* $(\phi \wedge (\psi \wedge \varphi)) \vdash ((\phi \wedge \psi) \wedge \varphi)$*:*

$$\dfrac{\dfrac{(\wedge_e)\,\dfrac{(\phi \wedge (\psi \wedge \varphi))}{\phi} \qquad \dfrac{\dfrac{(\phi \wedge (\psi \wedge \varphi))}{(\psi \wedge \varphi)}\,(\wedge_e)}{\psi}\,(\wedge_e)}{(\phi \wedge \psi)}\,(\wedge_i) \qquad \dfrac{\dfrac{(\phi \wedge (\psi \wedge \varphi))}{(\psi \wedge \varphi)}\,(\wedge_e)}{\varphi}\,(\wedge_e)}{((\phi \wedge \psi) \wedge \varphi)}\,(\wedge_i)$$

Exercise 7. *As an exercise, prove that* $((\phi \wedge \psi) \wedge \varphi) \vdash (\phi \wedge (\psi \wedge \varphi))$.

- *Second, the associativity of disjunction is proved; that is,* $(\phi \vee (\psi \vee \varphi)) \vdash ((\phi \vee \psi) \vee \varphi)$*:*

$$\dfrac{(\phi \vee (\psi \vee \varphi)) \qquad (\vee_i)\,\dfrac{(\vee_i)\,\dfrac{[\phi]^x}{(\phi \vee \psi)}}{((\phi \vee \psi) \vee \varphi)} \qquad \begin{array}{c}\nabla \\ ((\phi \vee \psi) \vee \varphi)\end{array}}{((\phi \vee \psi) \vee \varphi)}\,(\vee_e)\ x, y$$

where ∇ *is the derivation below:*

$$\dfrac{[(\psi \vee \varphi)]^y \qquad (\vee_i)\,\dfrac{(\vee_i)\,\dfrac{[\psi]^u}{(\phi \vee \psi)}}{((\phi \vee \psi) \vee \varphi)} \qquad \dfrac{[\varphi]^u}{((\phi \vee \psi) \vee \varphi)}\,(\vee_i)}{((\phi \vee \psi) \vee \varphi)}\,(\vee_e)\ u, v$$

Exercise 8. *As an exercise, prove that* $((\phi \vee \psi) \vee \varphi) \vdash (\phi \vee (\psi \vee \varphi))$.

Exercise 9. *Classify the derived rules of Table 1.4 discriminating those that belong to the intuitionistic fragment of propositional logic, and those that are classical. For instance,* (CP_1) *was proved above using only intuitionistic rules which means that it belongs to the intuitionistic fragment.*
Hint: to prove that a derived rule is not intuitionistic, one can show that using only intuitionistic rules and the derived rule a strictly classical rule such as (PBC) , (LEM) *or* $(\neg\neg_e)$ *can be derived.*

Exercise 10. *Check whether each variant of contraposition below is either an intuitionistic or a classical rule.*

$$\dfrac{\neg\varphi \to \psi}{\neg\psi \to \varphi}\,(\mathrm{CP}_3) \qquad\qquad \dfrac{\varphi \to \neg\psi}{\psi \to \neg\varphi}\,(\mathrm{CP}_4)$$

Exercise 11. *Similarly, check whether each variant of* (MT) below is either an intuitionistic or a classical rule.

$$\dfrac{\varphi \to \neg\psi \qquad \psi}{\neg\varphi}\,(\mathrm{MT}_2) \qquad \dfrac{\neg\varphi \to \psi \qquad \neg\psi}{\varphi}\,(\mathrm{MT}_3) \qquad \dfrac{\neg\varphi \to \psi \qquad \neg\psi}{\varphi}\,(\mathrm{MT}_4)$$

Exercise 12. *Using only the rules for the minimal propositional calculus, i.e. the rules in Table 1.2 without* $(\bot_e)$*, give derivations for the following sequents.*

a. $\neg\neg\neg\phi \dashv\vdash \neg\phi$.

b. $\neg\neg(\phi \to \psi) \vdash (\neg\neg\phi) \to (\neg\neg\psi)$.

c. $\neg\neg(\phi \wedge \psi) \dashv\vdash (\neg\neg\phi) \wedge (\neg\neg\psi)$.

d. $\neg(\phi \vee \psi) \dashv\vdash (\neg\phi \wedge \neg\psi)$.

e. $\phi \vee \psi \vdash \neg(\neg\phi \wedge \neg\psi)$.

f. $\vdash \neg\neg(\phi \vee \neg\phi)$.

Exercise 13. *Using the rules for the intuitionistic propositional calculus, that is the rules in Table 1.2, give derivations for the following sequents.*

a. $(\neg\neg\phi) \to (\neg\neg\psi) \vdash \neg\neg(\phi \to \psi)$. *Compare with item b of Exercise 12.*

b. $\vdash \neg\neg(\neg\neg\phi \to \phi)$.

Exercise 14. (*) *A propositional formula* ϕ *belongs to the negative fragment if it does not contain disjunctions and all propositional variables occurring in* ϕ *are preceded by negation. Formulas in this fragment have the following syntax.*

$$\phi ::= (\neg v) \;\; || \;\; \bot \;\; || \;\; (\neg\phi) \;\; || \;\; (\phi \wedge \phi) \;\; || \;\; (\phi \to \phi), \quad \textit{for } v \in V$$

Prove by induction on ϕ*, that for any formula in the negative fragment there are derivations in the minimal propositional calculus for*

$$\vdash \phi \leftrightarrow \neg\neg\phi$$

i.e. prove $\vdash \phi \to \neg\neg\phi$ *and* $\vdash \neg\neg\phi \to \phi$.

Exercise 15. *Give deductions for the following sequents:*

a. $\neg(\neg\phi \wedge \neg\psi) \vdash \phi \vee \psi$.

b. Peirce's law: $\vdash ((\phi \to \psi) \to \phi) \to \phi$.

Exercise 16. (*) *Let* Γ *be a set, and* φ *be a formula of propositional logic. Prove that if* φ *has a classical proof from the assumptions in* Γ *then* $\neg\neg\varphi$ *has an intuitionistic proof from the same assumptions. This fact is known as Glivenko's theorem (1929).*

Exercise 17. (*) *Consider the negative Gödel translation from classical propositional logic to intuitionistic propositional logic given by:*

- $\bot^n = \bot$
- $p^n = \neg\neg p$, *if* p *is a propositional variable.*
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \vee \psi)^n = \neg\neg(\varphi^n \vee \psi^n)$
- $(\varphi \to \psi)^n = \varphi^n \to \psi^n$

Prove that if $\Gamma \vdash \varphi$ *in classical propositional logic then* $\Gamma^n \vdash \varphi^n$ *in intuitionistic propositional logic.*

Exercise 18. *Prove the following sequent, the double negation of Peirce's law, in the intuitionistic propositional logic:* $\vdash \neg\neg(((\phi \to \psi) \to \phi) \to \phi)$

1.5 Semantics of the Propositional Logic

Deduction and derivation correspond to mechanical inference of *truth*. All syntactic deductive mechanisms that we have seen in the previous section can be blindly followed in order to prove that a formula of the propositional logic "holds", but in fact there was not presented a semantical counterpart of the notion of *being provable*. In this section we present the simple semantics of propositional logic.

In propositional logic the two only possible *truth-values* are *True* and *False*, denoted by brevity as T and F. No other truth-values are admissible, as it is the case in several other logical systems (e.g., truth-values as *may be true*, *probably*, *don't know*, *almost true*, *not yet, but in the future*, etc.).

Definition 9 (Truth values of atomic formula and assignments)**.** *In propositional logic the truth-values of the basic syntactic formula, that are $\bot$, $\top$ and variables in V, are given in the following manner:*

- *the truth-value of $\bot$ is F;*
- *the truth-value of $\top$ is T;*
- *the truth-value of a variable v in the set of variables V, is given trough a propositional assignment function from V to $\{T, F\}$. Thus, given an assignment function $d : V \to \{T, F\}$, the truth-value of $v \in V$ is given by $d(v)$.*

The truth-value assignment to propositional variables deserve special attention. Firstly, an assignment is necessary, because variables neither can be interpreted as true or false without having fixed an assignment. Secondly, only after one has an assignment, it is possible to decide whether (the truth-value of) a variable is either true or false. Finally, the true-value of propositional variables exclusively depends of a unique given assignment function.

Once an assignment function is given, one can determine the truth-value or semantical interpretation of non atomic propositional formulas according to the following inductive definition.

Definition 10 (Interpretation of propositional formula)**.** *Given an assignment d over the set of variables V, the truth-value or interpretation of a propositional formula φ is determined inductively as below:*

i. If $\varphi = \bot$ or $\varphi = \top$, one says that φ is F or T, respectively;

ii. if $\varphi = v \in V$, one says that φ is $d(v)$;

iii. if $\varphi = (\neg\psi)$*, then its interpretation is given from the interpretation of* ψ *by the truth-table below:*

ψ	$\varphi = (\neg\psi)$
T	F
F	T

iv. if $\varphi = (\psi \vee \phi)$*, then its interpretation is given from the interpretations of* ψ *and* ϕ *according to the truth-table below:*

ψ	ϕ	$\varphi = (\psi \vee \phi)$
T	T	T
T	F	T
F	T	T
F	F	F

v. if $\varphi = (\psi \wedge \phi)$*, then its interpretation is given from the interpretations of* ψ *and* ϕ *according to the truth-table below:*

ψ	ϕ	$\varphi = (\psi \wedge \phi)$
T	T	T
T	F	F
F	T	F
F	F	F

vi. if $\varphi = (\psi \rightarrow \phi)$*, then its interpretation is given from the interpretations of* ψ *and* ϕ

according to the truth-table below:

ψ	ϕ	$\varphi = (\psi \to \phi)$
T	T	T
T	F	F
F	T	T
F	F	T

According to this definition, it is possible to determine *the* truth-value of any propositional formula under a specific assignment. For instance, to determine that the formula $(v \to (\neg v))$ is false for a given assignment d for which $d(v) = T$, one can build the following *truth-table* according to the assignment of v under d and the inductive steps for the connectives $\neg$ and $\to$ of the definition:

v	$(\neg v)$	$(v \to (\neg v))$
T	F	F

Similarly, if d' is an assignment for which, $d'(v) = F$, one obtains the following *truth-table*:

v	$(\neg v)$	$(v \to (\neg v))$
F	T	T

Notice, that *the* interpretation of a formula depends on the given assignment. Also, although we are talking about *the* interpretation of a formula under a given assignment it was not proved that, given an assignment, formulas have a unique interpretation. That is done in the following lemma.

Lemma 1 (Uniqueness of interpretations). *The interpretation of a propositional formula φ under a given assignment d is unique and it is either true or false.*

Proof. The proof is by induction on the structure of propositional formulas.
IB In the three possible cases the truth-value is unique: for $\bot$ false, for $\top$ true and for $v \in V$, $d(v)$ that is unique since d is functional.

IS This is done by cases.

Case $\varphi = (\neg\psi)$. By the hypothesis of induction ψ is either true or false and consequently, following the item *iii.* of the definition of interpretation of propositional formulas, the interpretation of φ is univocally given by either false or true, respectively.

Case $\varphi = (\psi \vee \phi)$. By the hypothesis of induction the truth-values of ψ and ϕ are unique and consequently, according to the item *iv.* of the definition of interpretation of propositional formulas, the truth-value of φ is unique.

Case $\varphi = (\psi \wedge \phi)$. By the hypothesis of induction the truth-values of ψ and ϕ are unique and consequently, according to the item *v.* of the definition of interpretation of propositional formulas, the truth-value of φ is unique.

Case $\varphi = (\psi \rightarrow \phi)$. By the hypothesis of induction the truth-values of ψ and ϕ are unique and consequently, according to the item *vi.* of the definition of interpretation of propositional formulas, the truth-value of φ is unique.

□

It should be noticed that a formula may be interpreted both as true and false for different assignments. Uniqueness of the interpretation of a formula holds only once an assignment is fixed. Notice, for instance that the formula $(v \rightarrow (\neg v))$ can be true or false, according to the selected assignment. If it maps v to T, the formula is false and in the case that it maps v to F, the formula is true.

Whenever a formula can be interpreted as true for some assignment, it is said that the formula is satisfiable. In the other case it is said that the formula is unsatisfiable or invalid.

Definition 11 (Satisfiability and unsatisfiability)**.** *Let φ be a propositional formula. If there exists an assignment d, such that φ is true under d, then it is said to be satisfiable. If there does not exist such an assignment, it is said that φ is unsatisfiable.*

The semantical counterpart of derivability is the notion of being a *logical consequence.*

Definition 12 (Logical consequence and validity)**.** *Let, $\Gamma = \{\phi_1, \ldots, \phi_n\}$ be a finite set of propositional formulas that can be empty, and φ be a propositional formula. Whenever for* *all assignments under which all formulas of* Γ *are true, also* φ *is true, one says that* φ *is a logical consequence of* Γ*, which is denoted as*

$$\Gamma \models \varphi$$

When Γ *is the empty set one says that* φ *is valid, which is denoted as*

$$\models \varphi$$

Notice that the notion of validity of a propositional formula φ, corresponds to the nonexistence of assignments for which φ is false. Then by simple observations of the definitions, we have the following lemma.

Lemma 2 (Satisfiability versus validity)**.**

i. Any valid formula is satisfiable.

ii. The negation of a valid formula is unsatisfiable

Proof. i. Let φ be a propositional formula such that $\models \varphi$. Then given any assignment d, φ is true under d. Thus, φ is satisfiable.

ii. Let φ be a formula such that $\models \varphi$. Then for all assignments φ is true, which implies that for all assignments $(\neg\varphi)$ is false. Then there is no possible assignment for which $(\neg\varphi)$ is true. Thus, $(\neg\varphi)$ is unsatisfiable.

□

1.6 Soundness and Completeness of the Propositional Logic

The notions of *soundness* (or *correctness*) and *completeness* are not restricted to deductive systems being also applied in several areas of computer science. For instance, we can say