1.6.2 Completeness of the Propositional Logic

Now, we will prove that the propositional calculus, as given by the rules of natural deduction presented in Table 1.3, is also complete; that is, each logical consequence can be effectively proved through application of rules of the propositional logic. As a preliminary result, we will prove that each valid formula is in fact a formally provable theorem: $\models \varphi$ implies $\vdash \varphi$. Then, we will prove that this holds in general: whenever $\Gamma \models \varphi$, there exists a deduction for the sequent $\Gamma \vdash \varphi$, being Γ a finite set of propositional formulas.

To prove that validity implies provability, an auxiliary lemma is necessary.

Lemma 3 (Truth-values, assignments and deductions)**.** *Let V be a set of propositional variables, φ be a propositional formula containing only the propositional variables $v_1, \dots, v_n$ in V and let d be an assignment. Additionally, let $\widehat{v}^d$ denote the formula v whenever $d(v) = T$ and the formula $\neg v$, whenever $d(v) = F$, for $v \in V$. Then, one has*

- *If φ is true under assignment d, then*

$$\widehat{v_1}^d, \dots, \widehat{v_n}^d \vdash \varphi$$

- *Otherwise,*

$$\widehat{v_1}^d, \dots, \widehat{v_n}^d \vdash \neg\varphi$$

Proof. The proof is by induction on the structure of φ.

IB. The three possible cases are easily verified:

Case $\bot$ for $\varphi = \bot$, $\vdash \neg\bot$;

Case $\top$ for $\varphi = \top$, $\vdash \top$;

Case variable for $\varphi = v \in V$, since φ contains only variables in $v_1, \dots, v_n$, then $\varphi = v_i$, for some $1 \leq i \leq n$. Two possibilities should be considered: if $d(v_i) = T$, one has $\widehat{v_i}^d \vdash v_i$, that is $v_i \vdash v_i$; if $d(v_i) = F$, one has $\widehat{v_i}^d \vdash \neg v_i$, that is $\neg v_i \vdash \neg v_i$.

IS. The analysis proceeds by cases according to the structure of φ.

Case $\varphi = (\neg\psi)$. Observe that the set of variables occurring in φ and ψ is the same. In addition, by the semantics of negation, when φ is true under assignment d, ψ should be false and ψ is true under assignment d only if φ is false under this assignment.

By induction hypothesis, whenever ψ is false under assignment d it holds that

$$\widehat{v_1}^d, \dots, \widehat{v_n}^d \vdash (\neg\psi) = \varphi$$

that is what we need to prove in this case in which φ is true. Also, by induction hypothesis, whenever ψ is true under assignment d one has

$$\widehat{v_1}^d, \dots, \widehat{v_n}^d \vdash \psi$$

which means that there is a deduction of ψ from the formulas $\widehat{v_1}^d, \dots, \widehat{v_n}^d$. Thus, also a proof from this set of formulas of $\neg\varphi$ is obtained as below.

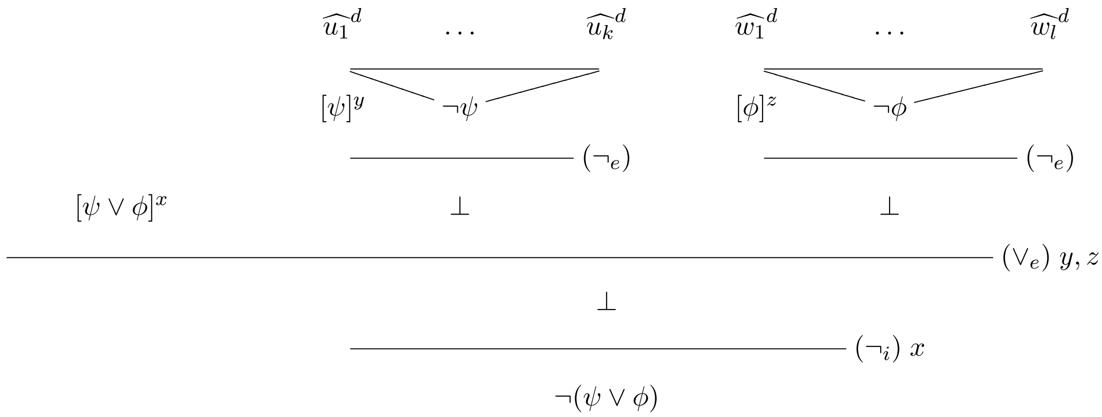
$$\dfrac{\begin{array}{c}\widehat{v_1}^d \quad \dots \quad \widehat{v_n}^d \\ \hline \searrow \ \psi \ \swarrow \end{array}}{(\neg\neg\psi)}\ (\neg\neg_i)$$

For the cases in which φ is a conjunction, disjunction or implication, of formulas ψ and ϕ, we will use the following notational convention: $\{u_1, \dots, u_k\}$ and $\{w_1, \dots, w_l\}$ are the sets of variables occurring in the formulas ψ and ϕ. Observe that these sets are not necessarily disjoint and that their union will give the set of variables $\{v_1, \dots, v_n\}$ occurring in φ.

Case $\varphi = (\psi \vee \phi)$. On the one side, suppose, φ is false under assignment d. Then, by the semantics of disjunction, both ψ and ϕ are false too, and by induction hypothesis, there are proofs for the sequents

$$\widehat{u_1}^d, \dots, \widehat{u_k}^d \vdash \neg\psi \quad \text{and} \quad \widehat{w_1}^d, \dots, \widehat{w_l}^d \vdash \neg\phi$$

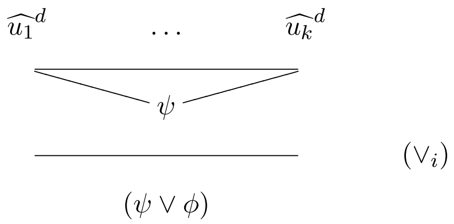
Thus, a proof of $\neg\varphi$, that is $\neg(\psi \vee \phi)$, is obtained combining proofs for these sequents as follows.



On the other side, suppose that φ is true. Then, by the semantics of disjunction, either ψ or ϕ should be true under assignment d (both formulas can be true too). Suppose ψ is true, then by induction hypothesis, we have a derivation for the sequent

$$\widehat{u_1}^d, \ldots, \widehat{u_k}^d \vdash \psi$$

Using this proof we can obtain a proof of the sequent $\widehat{u_1}^d, \ldots, \widehat{u_k}^d \vdash \varphi$, which implies that the desired sequent also holds: $\widehat{v_1}^d, \ldots, \widehat{v_n}^d \vdash \varphi$. The proof is depicted below.



The case in which ψ is false and ϕ is true is done in the same manner, adding an application of rule $(\vee_i)$ at the root of the derivation for the sequent

$$\widehat{w_1}^d, \ldots, \widehat{w_l}^d \vdash \phi$$

Case $\varphi = (\psi \wedge \phi)$. On the one side, suppose, φ is true under assignment d. Then, by the semantics of disjunction, both ψ and ϕ are true too, and by induction hypothesis, there are proofs for the sequents

$$\widehat{u_1}^d, \dots, \widehat{u_k}^d \vdash \psi \text{ and } \widehat{w_1}^d, \dots, \widehat{w_l}^d \vdash \phi$$

Thus, a proof of φ, that is $(\psi \wedge \phi)$, is obtained combining proofs for these sequents as follows.

$$\dfrac{\dfrac{\widehat{u_1}^d \quad \dots \quad \widehat{u_k}^d}{\psi} \qquad \dfrac{\widehat{w_1}^d \quad \dots \quad \widehat{w_l}^d}{\phi}}{\psi \wedge \phi}\,(\wedge_i)$$

On the other side, suppose that φ is false under assignment d. Then, some of the formulas ψ or ϕ should be false, by the semantical interpretation of conjunction. Suppose that ψ is false. The case in which ϕ is false is analogous. Then, by induction hypothesis, one has a derivation for the sequent

$$\widehat{u_1}^d, \dots, \widehat{u_k}^d \vdash \neg\psi$$

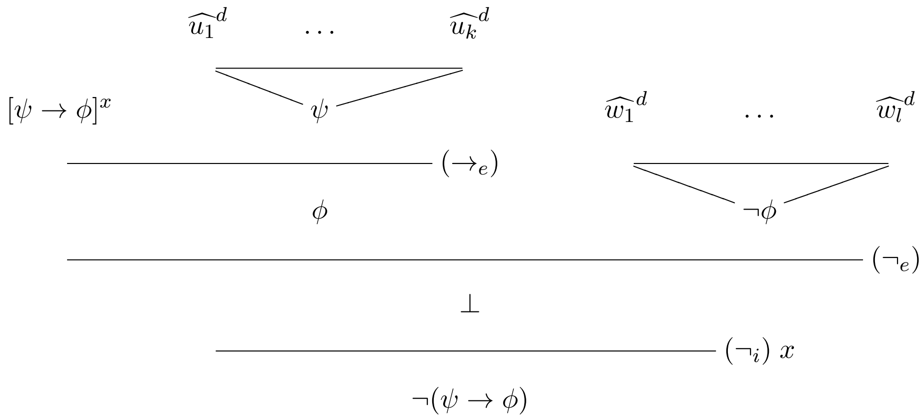
and the derivation for $\neg(\psi \wedge \phi)$, that is for φ, is obtained as depicted below.

$$\dfrac{\dfrac{\dfrac{\widehat{u_1}^d \quad \dots \quad \widehat{u_k}^d}{\neg\psi} \qquad \dfrac{[\psi \wedge \phi]^x}{\psi}\,(\wedge_e)}{\bot}\,(\neg_e)}{\neg(\psi \wedge \phi)}\,(\neg_i)\ x$$

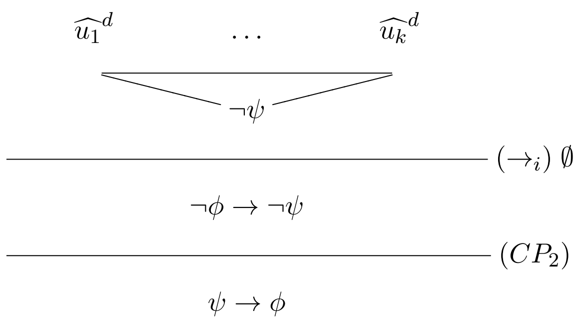
Case $\varphi = (\psi \to \phi)$. On the one side, suppose, φ is false under assignment d. Then, by the semantics of implication, ψ is true and ϕ false, and by induction hypothesis, there are proofs for the sequents

$$\widehat{u_1}^d, \ldots, \widehat{u_k}^d \vdash \psi \quad \text{and} \quad \widehat{w_1}^d, \ldots, \widehat{w_l}^d \vdash \neg\phi$$

Thus, a proof of $\neg\varphi$, that is $\neg(\psi \to \phi)$, is obtained combining proofs for these sequents as follows.



On the other side, if φ is true under assignment d, two cases should be considered according to the semantics of implication. Firstly, if ϕ is true, a proof can be obtained from the one for the sequent $\widehat{w_1}^d, \ldots, \widehat{w_l}^d \vdash \phi$, adding an application of rule $(\to_i)$ discharging an empty set of assumptions for ψ and concluding $\psi \to \phi$. Secondly, if ψ is false, a derivation can be built from the proof for the sequent $\widehat{u_1}^d, \ldots, \widehat{u_k}^d \vdash \neg\psi$ as depicted below.



□

Corollary 1 (Validity and provability for propositional formulas without variables)**.** *Suppose* $\models \varphi$*, for a formula* φ *without occurrences of variables. Then,* $\vdash \varphi$*.*

Exercise 19. *Prove the previous corollary.*

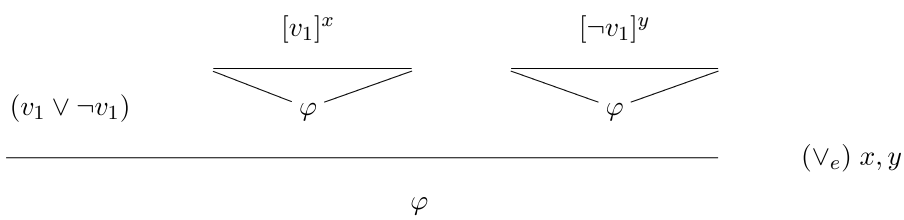
Theorem 4 (Completeness: validity implies provability)**.** *For all formula of the propositional logic*

$$\models \varphi \textit{ implies } \vdash \varphi$$

Proof. (Sketch) The proof is by an inductive argument on the variables occurring in φ: in each step of the inductive analysis we will get rid of the assumptions in the derivations of φ (built accordingly to Lemma 3) related with one variable of the initial set. Thus, the induction is specifically in the number of variables in φ minus the number of variables that are been eliminated from the assumptions until the current step of the process. In the end, a derivation for $\vdash \varphi$ without any assumption will be reached.

Suppose one has n variables occurring in φ, say $\{v_1, \ldots, v_n\}$. By the construction of the previous lemma, since $\models \varphi$, one has proofs for all of the 2^n possible designations for the n variables. Selecting a variable v_n one will have 2^{n-1} different proofs of φ with assumption v_n and other 2^{n-1} different proofs with assumption $\neg v_n$. Assembling these proofs with applications of (LEM) (for all formulas $v_i \vee \neg v_i$, for $i \neq n$) and rule $(\vee_e)$, as illustrated below, one obtains a derivation for $v_n \vdash \varphi$ and $\neg v_n \vdash \varphi$, from which a proof for $\vdash \varphi$ is also obtained using (LEM) (for $v_n \vee \neg v_n$) and $(\vee_e)$. The inductive sketch of the proof is as follows.

IB. The case in which φ has no occurrences of variables holds by the Corollary 1. Consider φ has only one variable v_1, Then by a simple application of rule $(\vee_e)$, proofs for $v_1 \vdash \varphi$ and $\neg v_1 \vdash \varphi$, are assembled as below obtaining a derivation for $\vdash \varphi$. The existence of proofs for $v_1 \vdash \varphi$ and $\neg v_1 \vdash \varphi$ is guaranteed by Lemma 3.



IS. Suppose φ has $n > 1$ variables. Since $\models \varphi$, by Lemma 3 one has 2^{n-1} different derivations for $v_n, \widehat{v_1}^d, \dots, \widehat{v_{n-1}}^d \vdash \varphi$ as well for $\neg v_n, \widehat{v_1}^d, \dots, \widehat{v_n}^d \vdash \varphi$, for all possible designations d. To get rid of the variable v_n one can use these derivations and (LEM) as below.

$$\dfrac{(v_n \vee \neg v_n) \qquad \begin{array}{c}[\widehat{v_1}^d]\dots[\widehat{v_{n-1}}^d][v_n]^x\\ \hline \vdots \\ \varphi\end{array} \qquad \begin{array}{c}[\widehat{v_1}^d]\dots[\widehat{v_{n-1}}^d][\neg v_n]^y\\ \hline \vdots \\ \varphi\end{array}}{\varphi}\ (\vee_e)\ x,y$$

In this manner, one builds, for each variable assignment d, a derivation for $\widehat{v_1}^d, \dots, \widehat{v_{n-1}}^d \vdash \varphi$. Proceeding in this way, that is using (LEM) for other variables and assembling the proofs using the rule $(\vee_e)$ one will be able to get rid of all other variables until a derivation for $\vdash \varphi$ is obtained.

To let things clearer to the reader, notice that the first step analyzed above implies that there are derivations ∇ and ∇' respectively for the sequents $\widehat{v_1}^d, \dots, \widehat{v_{n-2}}^d, v_{n-1} \vdash \varphi$ and $\widehat{v_1}^d, \dots, \widehat{v_{n-2}}^d, \neg v_{n-1} \vdash \varphi$. This is possible since in the previous analysis the assignment d is arbitrary; then, derivations as the one depicted above exist for assignments that map v_n either to true or false. Thus, a derivation for $\widehat{v_1}^d, \dots, \widehat{v_{n-2}}^d \vdash \varphi$ is obtained using (LEM) for the formula $v_{n-1} \vee \neg v_{n-1}$, the derivations ∇ and ∇', and the rule $(\vee_e)$, that will discharge the assumptions $[v_{n-1}]$ and $[\neg v_{n-1}]$ in the derivations ∇ and ∇', respectively.

□

Remark 3. *To clarify the way in which derivations are assembled in the previous inductive proof, let consider the case of a valid formula φ with three propositional variables p, q and r and for brevity let $\nabla_{000}, \nabla_{001}, \dots, \nabla_{111}$, denote derivations for $p, q, r \vdash \varphi$; $p, q, \neg r \vdash \varphi$; $\dots$, $\neg p, \neg q, \neg r \vdash \varphi$, respectively. Notice that the existence of derivations ∇_{ijk}, for $i, j, k = \{0, 1\}$ is guaranteed by Lemma 3.*

Derivations, ∇_{00} for $p, q \vdash \varphi$ and ∇_{01} for $p, \neg q \vdash \varphi$ are obtained as illustrated below.

$$\nabla_{00}: \qquad \dfrac{r \vee \neg r \qquad \begin{array}{c}[p]^x[q]^y[r]^z\\ \nabla_{000}\\ \varphi\end{array} \qquad \begin{array}{c}[p]^x[q]^y[\neg r]^{z'}\\ \nabla_{001}\\ \varphi\end{array}}{\varphi}\ (\vee_e)\ z, z'$$

$$\nabla_{01}: \qquad \dfrac{r \vee \neg r \qquad \begin{array}{c}[p]^x[\neg q]^{y'}[r]^z\\ \nabla_{010}\\ \varphi\end{array} \qquad \begin{array}{c}[p]^x[\neg q]^{y'}[\neg r]^{z'}\\ \nabla_{011}\\ \varphi\end{array}}{\varphi}\ (\vee_e)\ z, z'$$

Combining the two previous derivations, a proof ∇_0 *is obtained for* $p \vdash \varphi$ *as follows.*

$$\nabla_0: \qquad \dfrac{q \vee \neg q \qquad \begin{array}{c}[p]^x[q]^y\\ \nabla_{00}\\ \varphi\end{array} \qquad \begin{array}{c}[p]^x[\neg q]^{y'}\\ \nabla_{01}\\ \varphi\end{array}}{\varphi}\ (\vee_e)\ y, y'$$

Analogously, combining proofs ∇_{100} *and* ∇_{101} *one obtains derivations* ∇_{10} *and* ∇_{11} *respectively for* $\neg p, q \vdash \varphi$ *and* $\neg p, \neg q \vdash \varphi$. *From These two derivations it's possible to build a derivation* ∇_1 *for* $\neg p \vdash \varphi$. *Finally, from* ∇_0 *and* ∇_1, *proofs for* $p \vdash \varphi$ *and* $\neg p \vdash \varphi$, *one obtains the desired derivation for* $\vdash \varphi$.

The whole assemble, that is a derivation ∇ *for* $\vdash \varphi$, *is depicted below. Notice the drawback of being exponential in the number of variables occurring in the valid formula* φ.

$$\nabla\ \dfrac{\nabla_0\ \dfrac{\nabla_{00}\ \dfrac{\begin{array}{c}\nabla_{000}\\ \varphi\end{array}\ \begin{array}{c}\nabla_{001}\\ \varphi\end{array}}{\varphi} \qquad \nabla_{01}\ \dfrac{\begin{array}{c}\nabla_{010}\\ \varphi\end{array}\ \begin{array}{c}\nabla_{011}\\ \varphi\end{array}}{\varphi}}{\varphi} \qquad \nabla_1\ \dfrac{\nabla_{10}\ \dfrac{\begin{array}{c}\nabla_{100}\\ \varphi\end{array}\ \begin{array}{c}\nabla_{101}\\ \varphi\end{array}}{\varphi} \qquad \nabla_{11}\ \dfrac{\begin{array}{c}\nabla_{110}\\ \varphi\end{array}\ \begin{array}{c}\nabla_{111}\\ \varphi\end{array}}{\varphi}}{\varphi}}{\varphi}$$

Exercise 20. *Build a derivation for the instance of Peirce's law in propositional variables* p

and q according to the inductive construction of the proof of the completeness (Theorem 4). That is, first build derivations for $p, q \vdash ((p \to q) \to p) \to p$, $p, \neg q \vdash ((p \to q) \to p) \to p$, $\neg p, q \vdash ((p \to q) \to p) \to p$ and $\neg p, \neg q \vdash ((p \to q) \to p) \to p$, and then assemble these proofs to obtain a derivation for $\vdash ((p \to q) \to p) \to p$.

Finally, we proceed to prove the general version of the completeness of propositional logic, that is

$$\Gamma \models \varphi \quad \text{implies} \quad \Gamma \vdash \varphi$$

Theorem 5 (Completeness of Propositional Logic)**.** *Let Γ be a finite set of propositional formulas, and φ be a propositional formula. If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

Proof. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$. Initially, notice that

$$\gamma_1, \dots, \gamma_n \models \varphi \quad \text{implies} \quad \models \gamma_1 \to (\gamma_2 \to (\cdots (\gamma_n \to \varphi) \cdots))$$

Indeed, by contraposition, $\gamma_1 \to (\gamma_2 \to (\cdots (\gamma_n \to \varphi) \cdots))$ can only be false if all formulas γ_i, for $i = 1, ..., n$ are true and φ is false, which gives a contradiction to the assumption that φ is a logical consequence of Γ.

By, Theorem 4, the valid formula $\gamma_1 \to (\gamma_2 \to (\cdots (\gamma_n \to \varphi) \cdots))$ should be provable, that is, there exists a derivation, say ∇, for

$$\vdash \gamma_1 \to (\gamma_2 \to (\cdots (\gamma_n \to \varphi) \cdots))$$

To conclude, a derivation ∇' for $\gamma_1, \dots, \gamma_n \vdash \varphi$ can be built from the derivation ∇ by assuming $[\gamma_1]$, $[\gamma_2]$, etc and eliminating the premises of the implication γ_1, γ_2, etc by repeatedly applications the rule $(\to_e)$, as depicted below.

$$
\cfrac{[\gamma_n]^{u_n} \qquad \cfrac{\cfrac{[\gamma_2]^{u_2} \qquad \cfrac{[\gamma_1]^{u_1} \qquad \overset{\nabla}{\gamma_1 \to (\gamma_2 \to (\cdots(\gamma_n \to \varphi)\cdots))}}{\gamma_2 \to (\cdots(\gamma_n \to \varphi)\cdots)}\,(\to_e)}{\vdots}\,(\to_e)}{\gamma_n \to \varphi}}{\varphi}\,(\to_e)
$$

□

Additional Exercise 21. *As explained before, the classical propositional logic can be characterized by any of the equivalent rules* (PBC), $(\neg\neg_e)$ or (LEM). Show that Peirce's law is also equivalent to any of these rules. In other words, build intuitionistic proofs for the rules (PBC), $(\neg\neg_e)$ and (LEM) assuming the rule:

$$\frac{}{((\phi \to \psi) \to \phi) \to \phi}\ (\text{LP})$$

Next, prove (LP) in three different ways: each proof should be done in the intuitionistic logic assuming just one of (PBC), $(\neg\neg_e)$ and (LEM) at a time.

Additional Exercise 22. *Prove the following sequents:*

a. $\phi \to (\psi \to \gamma), \phi \to \psi \vdash \phi \to \gamma$

b. $(\phi \lor (\psi \to \phi)) \land \psi \vdash \phi$

c. $\phi \to \psi \vdash ((\phi \land \psi) \to \phi) \land (\phi \to (\phi \land \psi))$

d. $\vdash \psi \to (\phi \to (\phi \to (\psi \to \phi)))$