the sequent $\Gamma \vdash \exists_x \psi$; the later labels a subtree with open leaves in $\{\gamma_1, \dots, \gamma_n\} \cup \{\psi[x/x_0]\}$ and corresponds to a derivation for the sequent $\Gamma, \psi[x/x_0] \vdash \varphi$, where x_0 is a variable that does not occur free in $\Gamma \cup \{\varphi\}$, as depicted in the figure below:

$$\dfrac{\dfrac{\gamma_1 \quad \dots \quad \gamma_n}{\exists_x \psi} \qquad \dfrac{[\psi[x/x_0]]^u \ \gamma_1 \quad \dots \quad \gamma_n}{\varphi}}{\varphi}\ (\exists_e)\ u$$

By induction hypothesis, one has $\Gamma \models \exists_x \psi$ and $\Gamma, \psi[x/x_0] \models \varphi$. The first means that for any interpretation I such that $I \models \Gamma$, $I \models \exists_x \psi$. Thus, there exists some $a \in \mathtt{D}$, the domain of I, such that $I\frac{x}{a} \models \psi$. Notice also that since x_0 does not occur in Γ, one has that $I\frac{x_0}{a} \models \Gamma$. From the second, since $I\frac{x_0}{a} \models \Gamma, \psi[x/x_0]$, one has that $I\frac{x_0}{a} \models \varphi$. But, since x_0 does not occurs in φ, one concludes that $I \models \varphi$. $\square$

Exercise 30. *Complete all other cases of the proof of the Theorem 6 of soundness of predicate logic.*

2.5.2 Completeness of the Predicate Logic

The completeness proof for the predicate logic is not a direct extension of the completeness proof for the propositional logic. The completeness theorem was first proved by Kurt Gödel, and here we present the general idea of a proof due to Leon Albert Henkin (for nice complete presentations see references mentioned in the chapter on suggested readings).

The kernel of the proof is based on the fact that *every consistent set of formulas is satisfiable*, where consistency of the set Γ means that the absurd is not derivable from Γ:

Definition 28. *A set Γ of predicate formulas is consistent if not $\Gamma \vdash \bot$.*

Note that if we assume that **every consistent set is satisfiable** then the completeness can be easily obtained as follows:

Theorem 7 (Completeness)**.** *Let Γ be a set of predicate formulas. If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

Proof. We prove that not $\Gamma \vdash \varphi$ implies not $\Gamma \models \varphi$. From not $\Gamma \vdash \varphi$ one has that $\Gamma \cup \{\neg\varphi\}$ is consistent because if $\Gamma \cup \{\neg\varphi\}$ were inconsistent then $\Gamma \cup \{\neg\varphi\} \vdash \bot$ by definition, and one could prove φ as follows:

$$\cfrac{\begin{array}{c}\Gamma, [\neg\varphi]^a \\ \vdots \\ \bot\end{array}}{\varphi}\ (\text{PBC})\ a$$

Therefore, $\Gamma \vdash \varphi$, which contradicts the supposition that not $\Gamma \vdash \varphi$. Now, since $\Gamma \cup \{\neg\varphi\}$ is consistent, by the assumption that consistent sets are satisfiable, we have that $\Gamma \cup \{\neg\varphi\}$ is satisfiable. Therefore, we conclude that not $\Gamma \models \varphi$. □

Our goal from now on is to prove that **every consistent set of formulas is satisfiable**. The idea is, given a consistent set of predicate formulas Γ, to build a model I for Γ, and since the sole available information is its consistency, this must be done by purely syntactical means, that is by using the language to build the desired model.

The key concepts in Henkin's proof are the notion of *witnesses* of existential formulas and extension of consistent sets of formulas to *maximally consistent* sets.

Definition 29 (Witnesses and maximally consistency)**.** *Let Γ be a set of formulas*

Γ *contains witnesses if and only if for every formula of the form $\exists_x\varphi$ in Γ, there exists a term t such that $\Gamma \vdash \exists_x\varphi \to \varphi[x/t]$.*

Γ *is maximally consistent if and only if for each formula φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$.*

Notice that from the definition, for any possible extension of a maximally consistent set Γ, say Γ' such that $\Gamma \subseteq \Gamma'$, $\Gamma' = \Gamma$. Maximally consistent sets are also said to be *closed for negation.*

The proof is done in two steps, and uses the fact that every subset of a satisfiable set is also satisfiable:

1. every consistent set can be extended to a maximally consistent set containing witnesses;

2. every maximally consistent set containing witnesses has a model.

If Γ does not contain witnesses, these formulas can not be built in a straightforward manner, since one can not choose any arbitrary term t to be witness of the existential formula without changing the semantics. Nevertheless, any consistent set can be extended to another consistent set containing witnesses. The simplest case, is when the language is countable and the set Γ uses only a finite set of free variables, that is $\mathtt{fv}(\Gamma)$ is finite. Since the set of existential formulas is also countable and there are infinite unused variable (those that do not appear free in Γ). Then these variables can be used as witnesses without any conflict. The other cases are more elaborated and are left as research exercises to the reader (Exercises 32 and 33): the case in which the language is countable, but Γ uses infinitely many free variables and the case in which the language is not countable.

In the sequel we will treat the simplest case in which the set of constant, function and predicate symbols occurring in Γ is at most countable and there are only finitely many variables occurring in Γ. The next two lemmas complete the first part of the proof: a consistent set might be extended to a maximally consistent set with witnesses. This is done proving first how variables might be used to include witnesses and then how a consistent set with witnesses can be extended to a maximally consistent set.

Lemma 4 (Construction of witnesses). *Let Γ be a consistent set over a countable language such that $\mathtt{fv}(\Gamma)$ is finite. There exists an extension $\Gamma' \supseteq \Gamma$ over the same language, such that Γ' is consistent and contains witnesses.*

Proof. Let $\exists_{x_1}\varphi_1, \exists_{x_2}\varphi_2, \ldots$ be an enumeration of all the existential formulas built over the language. Let $y_1, y_2, \ldots$ be an enumeration of the variables not occurring free in Γ, and consider the formulas below, for $i > 0$:

$$(\exists_{x_i}\varphi_i) \to \varphi_i[x_i/y_i]$$

Let Γ_0 be defined as Γ, and Γ_n, for $n > 0$ be defined as below:

$$\Gamma_n = \Gamma_{n-1} \cup \{(\exists_{x_n}\varphi_n) \to \varphi_n[x_n/y_n]\}$$

We will prove the consistence of Γ' defined as $\Gamma' = \bigcup_{n\in\mathbb{N}} \Gamma_n$ by induction on n. The base case is trivial since Γ is consistent by hypothesis. For $k > 0$, suppose Γ_{k-1} is consistent, but Γ_k is not, i.e.

$$\Gamma_k = \Gamma_{k-1} \cup \{(\exists_{x_k}\varphi_k) \to \varphi_k[x_k/y_k]\} \vdash \bot \tag{2.1}$$

Now consider the following derivation:

$$\dfrac{(\text{LEM})\ (\exists_{x_k}\varphi_k) \vee \neg(\exists_{x_k}\varphi_k) \qquad \begin{array}{c}\Gamma_{k-1}\ [\exists_{x_k}\varphi_k]^a\\ \nabla_1\\ \bot\end{array} \qquad \begin{array}{c}\Gamma_{k-1}\ [\neg\exists_{x_k}\varphi_k]^b\\ \nabla_2\\ \bot\end{array}}{\bot}\ (\vee e)\, a\, b$$

where

$$\nabla_1 : \quad \dfrac{[\exists_{x_k}\varphi_k]^a \qquad \dfrac{\Gamma_{k-1} \qquad \dfrac{[\varphi_k[x_k/y_k]]^u}{\exists_{x_k}\varphi_k \to \varphi_k[x_k/y_k]}\ (\to i)\emptyset}{\bot}\ (2.1)}{\bot}\ (\exists e) u$$

and

$$\nabla_2 : \quad \dfrac{\Gamma_{k-1} \qquad \dfrac{\dfrac{[\neg\exists_{x_k}\varphi_k]^b}{\neg\varphi_k[x_k/y_k] \to \neg\exists_{x_k}\varphi_k}\ (\to i)\emptyset}{\exists_{x_k}\varphi_k \to \varphi_k[x_k/y_k]}\ (\text{CP})}{\bot}\ (2.1)$$

But this is a proof of $\Gamma_{k-1} \vdash \bot$ which contradicts the assumption that Γ_{k-1} is consistent.

Therefore, Γ_k is consistent. □

In the previous proof, note that if $\Gamma_{i-1} \vdash \exists_{x_i}\varphi_i$ then it must be the case that $\Gamma_i \vdash \varphi_i[x_i/y_i]$ in order to preserve the consistency. Therefore, $\varphi_i[x_i/y_i]$ might be added to the set of formulas, but not its negation, as will be seen in the further construction of maximally consistent sets.

Now we prove that every maximally consistent set containing witnesses has a model.

Lemma 5 (Lindenbaum)**.** *Each consistent set of formulas* Γ *over a countable language is contained in a maximally consistent set* Γ^* *over the same language.*

Proof. Let $\delta_1, \delta_2, \dots$ be an enumeration of the formulas built over the language. In order to build a consistent expansion of Γ we recursively define the family of indexed sets of formulas Γ_i as follows:

- $\Gamma_0 = \Gamma$
- $\Gamma_i = \begin{cases} \Gamma_{i-1} \cup \{\delta_i\}, & \text{if } \Gamma_{i-1} \cup \{\delta_i\} \text{ is consistent;} \\ \Gamma_{i-1}, & \text{otherwise.} \end{cases}$

Now let $\Gamma^* = \bigcup_{i \in \mathbb{N}} \Gamma_i$. We claim that Γ^* is maximally consistent. In fact, if Γ^* is not maximally consistent then there exists a formula $\gamma \notin \Gamma^*$ such that $\Gamma^* \cup \{\gamma\}$ is consistent. But by the above enumeration, there exists $k \geq 1$ such that $\gamma = \delta_k$, and since $\Gamma_{k-1} \cup \{\gamma\}$ should be consistent, $\delta_k \in \Gamma_{k+1}$. Hence $\delta_k = \gamma \in \Gamma^*$. □

From the previous lemmas (4 and 5), one has that every consistent set of formulas built over a countable set of symbols and with finitely many free variables can be extended to a maximally consistent set which contains witnesses. In this manner we complete the first step of the prove.

Now, we will complete the second step of the proof, that is that any maximally consistent set that contain witnesses is satisfiable. We start with two auxiliary definitional observations.

Lemma 6. *Let* Γ *be a maximally consistent set of formulas. Then for any formula* φ *either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$*.*

Lemma 7. *Let Γ be a maximally consistent set. For any formula φ, $\Gamma \vdash \varphi$ if, and only if $\varphi \in \Gamma$.*

Proof. Suppose $\Gamma \vdash \varphi$. From Lemma 6, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. If $\neg\varphi \in \Gamma$ then Γ would be inconsistent:

$$\dfrac{\begin{array}{c}\Gamma\\ \nabla\\ \varphi\end{array} \qquad \begin{array}{c}\Gamma\\ \nabla\\ \neg\varphi\end{array}}{\bot}\ (\neg_e)$$

Therefore, $\varphi \in \Gamma$. □

We now define a model, that is called the *algebra* or *structure of terms* for the set Γ which is assumed to be maximally consistent and containing witnesses. The model, denoted as I_Γ, is built from Γ by taking as domain, the set $\mathtt{D}$ of all terms built over the countable language of Γ as given in the definition of terms 13. The designation d for each variable is the same variable and the interpretation of each non variable term is itself too: $t^{I_\Gamma} = t$. Notice that since our predicate language does not deal with equality symbol, different terms are interpreted as different elements of $\mathtt{D}$. The map $\mathtt{m}$ of I_Γ maps each n-ary function symbol in the language, f, in the function f^{I_Γ} such that for all terms $t_1, \dots, t_n$, $(f(t_1, \dots, t_n))^{I_\Gamma} = f^{I_\Gamma}(t_1^{I_\Gamma}, \dots, t_n^{I_\Gamma}) = f(t_1, \dots, t_n)$, and for each n-ary predicate symbol p, p^{I_Γ} is the relation defined as

$$(p(t_1, \dots, t_n))^{I_\Gamma} = p^{I_\Gamma}(t_1^{I_\Gamma}, \dots, t_n^{I_\Gamma}) \text{ if and only if } p(t_1, \dots, t_n) \in \Gamma$$

With these definitions we have that for any atomic formula φ, $\varphi \in \Gamma$ if and only if $I_\Gamma \models \varphi$. In addition, according to the interpretation of quantifiers, for any atomic formula $\forall_{x_1} \dots \forall_{x_n}\varphi \in \Gamma$ if and only if $I_\Gamma \models \forall_{x_1} \dots \forall_{x_n}\varphi$ and $\exists_{x_1} \dots \exists_{x_n}\varphi \in \Gamma$ if and only if $I_\Gamma \models \exists_{x_1} \dots \exists_{x_n}\varphi$.

Using the assumptions that Γ has witnesses and is maximally consistent, formulas can be correctly interpreted in I_Γ as below.

1. $\bot^{I_\Gamma} = F$ and $\top^{I_\Gamma} = T$
2. $\varphi^{I_\Gamma} = T$, iff $\varphi \in \Gamma$, for any atomic formula φ
3. $(\neg\varphi)^{I_\Gamma} = T$, iff $\varphi^{I_\Gamma} = F$
4. $(\varphi \wedge \psi)^{I_\Gamma} = T$, iff $\varphi^{I_\Gamma} = T$ and $\psi^{I_\Gamma} = T$
5. $(\varphi \vee \psi)^{I_\Gamma} = T$, iff $\varphi^{I_\Gamma} = T$ or $\psi^{I_\Gamma} = T$
6. $(\varphi \to \psi)^{I_\Gamma} = T$, iff $\varphi^{I_\Gamma} = F$ or $\psi^{I_\Gamma} = T$
7. $(\exists_x \varphi)^{I_\Gamma} = T$, iff $(\varphi[x/t])^{I_\Gamma} = T$, for some term $t \in \mathtt{D}$
8. $(\forall_x \varphi)^{I_\Gamma} = T$, iff $(\varphi[x/t])^{I_\Gamma} = T$, for all $t \in \mathtt{D}$.

Indeed, this interpretation is well-defined only under the assumption that Γ has witnesses and is maximally consistent. For instance, the item 3 is well-defined since $\neg\varphi \in \Gamma$ if and only if not $\varphi \in \Gamma$. For the item 5, if $(\varphi \vee \psi) \in \Gamma$ and not $\varphi \in \Gamma$, by maximally consistency one has that $\neg\varphi \in \Gamma$; thus, from $(\varphi \vee \psi)$ and $\neg\varphi$, it is possible to derive ψ (by simple application of rules $(\vee_e)$ and $(\neg_e)$ and $(\bot_e)$). Similarly, if we assume $(\varphi \vee \psi) \in \Gamma$ and not $\psi \in \Gamma$, we can derive φ. For the item 6, suppose $(\varphi \to \psi) \in \Gamma$ and $\varphi \in \Gamma$, then one can derive ψ (by application of $(\to_e)$); otherwise, if $(\varphi \to \psi) \in \Gamma$ and not $\psi \in \Gamma$, by maximally consistency, $\neg\psi \in \Gamma$, from which one can infer $\neg\varphi$ (by application of contraposition). For the item 7, if we assume $\exists_x \varphi \in \Gamma$, by the existence of witnesses, there is a term t such that $\exists_x \varphi \to \varphi[x/t] \in \Gamma$, and from these two formulas we can derive $\varphi[x/t]$ (by a simple application of rule $(\to_e)$).

Exercise 31. *Complete the analysis well-definedness for all the items in the interpretation of formulas I_Γ, for a set Γ that contains witnesses and is maximally complete.*

Theorem 8 (Henkin)**.** *Let Γ be a maximally consistent set containing witnesses. Then for all φ,*

$$I_\Gamma \models \varphi, \text{ if, and only if } \Gamma \vdash \varphi.$$

Proof. The proof is done by induction on the structure of φ. If φ is an atomic formula then $\varphi \in \Gamma$ iff $(\varphi)^{I_\Gamma} = T$, by definition.

If $\varphi = \neg\varphi_1$ then:

$$\begin{array}{lcl}
\neg\varphi_1 \in \Gamma & \iff & \text{(because } \Gamma \text{ is maximally consistent)}\\
\varphi_1 \notin \Gamma & \iff & \text{(by induction hypothesis)}\\
\text{not } I_\Gamma \models \varphi_1 & \iff & \text{(by definition)}\\
I_\Gamma \models \neg\varphi_1. & &
\end{array}$$

If $\varphi = \varphi_1 \wedge \varphi_2$ then:

$$\begin{array}{lcl}
\varphi_1 \wedge \varphi_2 \in \Gamma & \iff & \text{(by definition)}\\
\varphi_1 \in \Gamma \text{ and } \varphi_2 \in \Gamma & \iff & \text{(by induction hypothesis for both } \varphi_1 \text{ and } \varphi_2)\\
I_\Gamma \models \varphi_1 \text{ and } I_\Gamma \models \varphi_2 & \iff & \text{(by definition)}\\
I_\Gamma \models \varphi_1 \wedge \varphi_2. & &
\end{array}$$

If $\varphi = \varphi_1 \vee \varphi_2$ then:

$$\begin{array}{lcl}
\varphi_1 \vee \varphi_2 \in \Gamma & \iff & \text{(by definition)}\\
\varphi_1 \in \Gamma \text{ or } \varphi_2 \in \Gamma & \iff & \text{(by induction hypothesis for both } \varphi_1 \text{ and } \varphi_2)\\
I_\Gamma \models \varphi_1 \text{ or } I_\Gamma \models \varphi_2 & \iff & \text{(by definition, no matter the condition holds for } \varphi_1 \text{ or } \varphi_2)\\
I_\Gamma \models \varphi_1 \vee \varphi_2. & &
\end{array}$$

If $\varphi = \varphi_1 \to \varphi_2$ then we split the proof into two parts. Firstly, we show that $\varphi_1 \to \varphi_2 \in \Gamma$ implies $I_\Gamma \models \varphi_1 \to \varphi_2$. We have two subcases:

1. $\varphi_1 \in \Gamma$: In this case, $\varphi_2 \in \Gamma$. In fact, if $\varphi_2 \notin \Gamma$ then $\neg\varphi_2 \in \Gamma$ by the maximality of Γ, and Γ becomes contradictorily inconsistent:

$$\dfrac{\dfrac{\varphi_1 \to \varphi_2 \qquad \varphi_1}{\varphi_2}\,(\to_e) \qquad \neg\varphi_2}{\bot}\,(\neg_e)$$

Thus, by induction hypothesis one has:

$$\begin{array}{lll} \varphi_1 \in \Gamma \text{ and } \varphi_2 \in \Gamma & \Longleftrightarrow & \text{(by induction hypothesis for both } \varphi_1 \text{ and } \varphi_2) \\ I_\Gamma \models \varphi_1 \text{ and } I_\Gamma \models \varphi_2 & \Longrightarrow & \text{(by definition)} \\ I_\Gamma \models \varphi_1 \to \varphi_2. & & \end{array}$$

2. $\varphi_1 \notin \Gamma$: In this case, $\neg\varphi_1 \in \Gamma$ by the maximality of Γ. Therefore,

$$\begin{array}{lll} \neg\varphi_1 \in \Gamma & \Longleftrightarrow & \text{(by induction hypothesis)} \\ I_\Gamma \models \neg\varphi_1 & \Longleftrightarrow & \text{(by definition)} \\ \text{not } I_\Gamma \models \varphi_1 & \Longrightarrow & \text{(by definition)} \\ I_\Gamma \models \varphi_1 \to \varphi_2. & & \end{array}$$

Now we prove that $I_\Gamma \models \varphi_1 \to \varphi_2$ implies $\varphi_1 \to \varphi_2 \in \Gamma$. By definition of the semantics of implication, there are two cases:

1. $\varphi_1^{I_\Gamma} = F$: In this case, we have that $(\neg\varphi_1)^{I_\Gamma} = T$, and hence $\neg\varphi_1 \in \Gamma$, by induction hypothesis. We can now derive $\varphi_1 \to \varphi_2$ as follows, and conclude by Lemma 7:

$$\dfrac{\dfrac{\dfrac{\neg\varphi_1 \qquad [\varphi_1]^a}{\bot}\,(\neg_e)}{\varphi_2}\,(\bot_e)}{\varphi_1 \to \varphi_2}\,(\to_i)\ a$$

2. $\varphi_2^{I_\Gamma} = T$: By induction hypothesis $\varphi_2 \in \Gamma$, and we derive $\varphi_1 \to \varphi_2$ as follows, and conclude by Lemma 7:

$$\dfrac{\varphi_2}{\varphi_1 \to \varphi_2}\,(\to_i)\ \emptyset$$

If $\varphi = \exists_x \varphi_1$ then:

$$\begin{array}{lll} \exists_x \varphi_1 \in \Gamma & \iff & \text{(for some } t \in \mathtt{D}\text{, since } \Gamma \text{ contains witnesses)} \\ \varphi_1[x/t] \in \Gamma & \iff & \text{(by induction hypothesis)} \\ I_\Gamma \models \varphi_1[x/t] & \iff & \text{(by definition)} \\ I_\Gamma \models \exists_x \varphi_1. & & \end{array}$$

If $\varphi = \forall_x \varphi_1$ then:

$$\begin{array}{lll} \forall_x \varphi_1 \in \Gamma & \iff & \text{(otherwise } \Gamma \text{ becomes inconsistent as shown below)} \\ \varphi_1[x/t] \in \Gamma, \text{ for all } t \in \mathtt{D} & \iff & \text{(by induction hypothesis)} \\ I_\Gamma \models \varphi_1[x/t], \text{ for all } t \in \mathtt{D} & \iff & \text{(by definition)} \\ I_\Gamma \models \forall_x \varphi_1. & & \end{array}$$

For the first equivalence, note that if $\forall_x \varphi_1 \in \Gamma$ then $\varphi_1[x/t] \in \Gamma$, for all term $t \in \mathtt{D}$, otherwise Γ becomes contradictorily inconsistent:

$$\dfrac{\neg\varphi_1[x/t] \qquad \dfrac{\forall_x \varphi_1}{\varphi_1[x/t]}\ (\forall_e)}{\bot}\ (\bot_e)$$

□

Using as a model I_Γ, it is possible to conclude, in this case, that consistent sets are satisfiable.

Corollary 2 (Consistency implies satisfiability)**.** *If* Γ *is a consistent set of formulas over a countable language with a finite set of free variables then* Γ *is satisfiable.*

Proof. Initially, Γ is consistently enlarged obtaining the set Γ' including witnesses according to the construction in Lemma 4; afterwards, Γ' is closed maximally obtaining the set $(\Gamma')^*$according to the construction in Lindenbaum's Lemma (5). This set contains witnesses and is maximally consistent; then, by Henkin's Theorem (8), I_Γ is a model of $(\Gamma')^*$, hence a model of Γ too. □

Exercise 32. (*) *Research in the suggested related references how a consistent set built over a countable set of symbols, but that uses infinite free variables can be extended to a maximal consistent set with witnesses. The problem, is that in this case there are no new variables that can be used as witnesses. Thus, one needs to extend the language with new constant symbols that will act as witnesses, but each time a new constant symbol is added to the language the set of existential formulas change.*

Exercise 33. (*) *Research the general case in which the language is not restricted, that is the case in which Γ is built over a non countable set of symbols.*

2.5.3 Compactness Theorem and Löwenheim-Skolem Theorem

The connections between $\models$ and $\vdash$ as well as between consistence and satisfiability provided in this section, give rise to other additional important consequences that relate semantic and syntactic elements of the predicate logic. Here we present two important theorems that are related with the scope and limits of the expressiveness of predicate logic.

Theorem 9 (Compactness)**.** *Given a set Γ of predicate formulas and a formula φ, the following holds*

i. $\Gamma \models \varphi$ *if and only if there is a finite set* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \models \varphi$

ii. Γ *is satisfiable if and only if for all finite set* $\Gamma_0 \subseteq \Gamma$, Γ_0 *is satisfiable.*

Proof. i. For necessity, if $\Gamma \models \varphi$, by completeness one has that there exists a derivation ∇ for $\Gamma \vdash \varphi$. The derivation ∇ uses only a finite subset of assumptions, say $\Gamma_0 \subseteq \Gamma$. Thus, $\Gamma_0 \vdash \varphi$ and, by correctness, one concludes that $\Gamma_0 \models \varphi$. For sufficiency, suppose that $\Gamma_0 \models \varphi$, for a finite set $\Gamma_0 \subseteq \Gamma$. By completeness there exists a derivation ∇ for $\Gamma_0 \vdash \varphi$. But ∇ is also a derivation for $\Gamma \vdash \varphi$; hence, by correctness one concludes that $\Gamma \models \varphi$.