2.5 Soundness and Completeness of the Predicate Logic

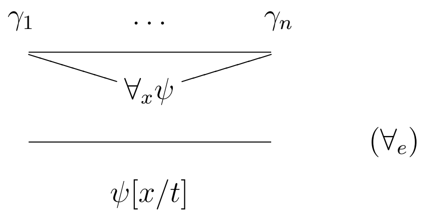
2.5.1 Soundness of the Predicate Logic

The soundness of predicate logic can be proved following the same idea used for the propositional logic. Therefore, we need to prove the following theorem:

Theorem 6 (Soundness of the predicate logic)**.** *Let* Γ *be a set of predicate formulas, if* $\Gamma \vdash \varphi$ *then* $\Gamma \models \varphi$*. In other words, if* φ *is provable from* Γ *then* φ *is a logical consequence of* Γ*.*

Proof. The proof is by induction on the derivation of $\Gamma \vdash \varphi$ similarly to the propositional case, and hence we focus just on the new rules: $(\forall_e)$, $(\forall_i)$, $(\exists_e)$, $(\exists_i)$.

If the last rule applied in the proof $\Gamma \vdash \varphi$ is $(\forall_e)$, then $\varphi = \psi[x/t]$ and the premise of the last rule is $\forall_x\psi$ as depicted in the following figure, where $\{\gamma_1, \ldots, \gamma_n\}$ is the subset of formulas in Γ used in the derivation.



The subtree rooted by the formula $\forall_x\psi$ and with open leaves labeled by formulas in Γ, corresponds to a derivation for the sequent $\Gamma \vdash \forall_x\psi$, that by induction hypothesis implies $\Gamma \models \forall_x\psi$. Therefore, for all interpretations that make the formulas in Γ true, also $\forall_x\psi$ would be true: $I \models \Gamma$ implies $I \models \forall_x\psi$. The last implies that for all $a \in \mathtt{D}$, where $\mathtt{D}$ is the domain of I, $I\frac{x}{a} \models \psi$, and in particular, $I\frac{x}{t^I} \models \psi$. Consequently, $I \models \psi[x/t]$. Therefore, one has that for any interpretation I, such that $I \models \Gamma$, $I \models \psi[x/t]$, which implies $\Gamma \models \psi[x/t]$.

If the last rule applied in the proof of $\Gamma \vdash \varphi$ is $(\forall_i)$, then $\varphi = \forall_x\psi$ and the premise of the last rule is $\psi[x/x_0]$ as depicted in the following figure:

$$\dfrac{\begin{array}{c}\gamma_1 \qquad \dots \qquad \gamma_n \\ \hline \psi[x/x_0]\end{array}}{\forall_x \psi}\ (\forall_i)$$

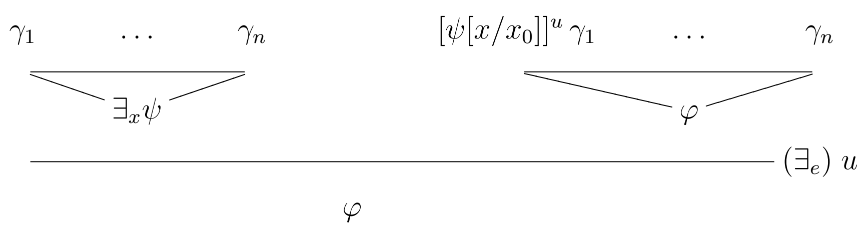
The subtree rooted by the formula $\psi[x/x_0]$ and with open leaves labeled by formulas in $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$, corresponds to a derivation for the sequent $\Gamma \vdash \psi[x/x_0]$, in which no open assumption contains the variable x_0. This variable can be selected in such a manner that it does not appear free in any formula of Γ. By induction hypothesis, we have that $\Gamma \models \psi[x/x_0]$. This implies that all interpretations that make the formulas in Γ true, also make $\psi[x/x_0]$ true: $I \models \Gamma$ implies $I \models \psi[x/x_0]$. Since x_0 does not occurs in Γ, for all $a \in \mathtt{D}$, where $\mathtt{D}$ is the domain of I, $I\frac{x}{a} \models \Gamma$ and also $I\frac{x_0}{a} \models \psi[x/x_0]$ or, equivalently, $I\frac{x}{a} \models \psi$. Hence $\Gamma \models \forall_x \psi$.

If the last rule applied in the proof of $\Gamma \vdash \varphi$ is $(\exists_i)$, then $\varphi = \exists_x \psi$ and the premise of the last rule is $\psi[x/t]$ as depicted in the following figure, where again $\{\gamma_1, \dots, \gamma_n\}$ is the subset of formulas of Γ used in the derivation:

$$\dfrac{\begin{array}{c}\gamma_1 \qquad \dots \qquad \gamma_n \\ \hline \psi[x/t]\end{array}}{\exists_x \psi}\ (\exists_i)$$

The subtree rooted by the formula $\psi[x/t]$ and with open leaves labeled by formulas of Γ, corresponds to a derivation of the sequent $\Gamma \vdash \psi[x/t]$, that by induction hypothesis implies $\Gamma \models \psi[x/t]$. Therefore, any interpretation I that makes the formulas in Γ true, also makes $\psi[x/t]$ true. Thus, since $I \models \psi[x/t]$ implies $I\frac{x}{t^I} \models \psi$, one has that $I \models \exists_x \psi$. Therefore, $\Gamma \models \exists_x \psi$.

Finally, for a derivation of the sequent $\Gamma \vdash \varphi$ that finishes with an application of the rule $(\exists_e)$, one has as premises the formulas $\exists_x \psi$ and φ. The former labels a root of a subtree with open leaves labeled by assumptions in $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ that corresponds to a derivation for the sequent $\Gamma \vdash \exists_x \psi$; the later labels a subtree with open leaves in $\{\gamma_1, \ldots, \gamma_n\} \cup \{\psi[x/x_0]\}$ and corresponds to a derivation for the sequent $\Gamma, \psi[x/x_0] \vdash \varphi$, where x_0 is a variable that does not occur free in $\Gamma \cup \{\varphi\}$, as depicted in the figure below:



By induction hypothesis, one has $\Gamma \models \exists_x \psi$ and $\Gamma, \psi[x/x_0] \models \varphi$. The first means that for any interpretation I such that $I \models \Gamma$, $I \models \exists_x \psi$. Thus, there exists some $a \in \mathtt{D}$, the domain of I, such that $I\frac{x}{a} \models \psi$. Notice also that since x_0 does not occur in Γ, one has that $I\frac{x_0}{a} \models \Gamma$. From the second, since $I\frac{x_0}{a} \models \Gamma, \psi[x/x_0]$, one has that $I\frac{x_0}{a} \models \varphi$. But, since x_0 does not occurs in φ, one concludes that $I \models \varphi$. $\square$

Exercise 30. *Complete all other cases of the proof of the Theorem 6 of soundness of predicate logic.*

2.5.2 Completeness of the Predicate Logic

The completeness proof for the predicate logic is not a direct extension of the completeness proof for the propositional logic. The completeness theorem was first proved by Kurt Gödel, and here we present the general idea of a proof due to Leon Albert Henkin (for nice complete presentations see references mentioned in the chapter on suggested readings).

The kernel of the proof is based on the fact that *every consistent set of formulas is satisfiable*, where consistency of the set Γ means that the absurd is not derivable from Γ:

Definition 28. *A set Γ of predicate formulas is consistent if not $\Gamma \vdash \bot$.*

Note that if we assume that **every consistent set is satisfiable** then the completeness can be easily obtained as follows:

Theorem 7 (Completeness)**.** *Let Γ be a set of predicate formulas. If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*