Chapter 3

Deductions in the Style of Gentzen's Sequent Calculus

In this chapter we present a style of deduction known as Gentzen's Sequent Calculus that is different from the one of Natural Deduction (both invented by Gerhard Gentzen) and that has relevant computational interest and applications. The goal of this section is to present the alternative of deduction *à la* Gentzen Sequent Calculus, proving its equivalence with Gentzen's natural deduction. This sequent style is the one used by the proof assistant PVS that will be used in the next chapter.

3.1 Motivation

Both deduction technologies, Natural Deduction and Gentzen's Sequent Calculus, were invented by the German mathematician Gerhard Gentzen in the 1930's, although it is known that the Polish logician Stanisław Jaśkowski was the first to present a system of natural deduction. In sequent calculi *à la* Gentzen (for short, we will use "calculus *à la* Gentzen" or "sequent calculus"), deductions are trees as in natural deduction, but instead formulas, nodes are labelled by *sequents* of the form:

$$\Gamma \Rightarrow \Delta$$

The sequent expresses that Δ is deducible from Γ, where Γ and Δ are sequences of formulas, or more precisely as we will see multisets indeed. The multiset Γ is called the *antecedent*, while Δ is the *succedent* of the sequent, or respectively, the premises and conclusions of the sequent.

From this point of view, Gentzen's sequent calculus can be interpreted as a meta calculus for systems of natural deduction. As a very simple example consider the sequent

$$\varphi \Rightarrow \varphi$$

According to the above interpretation, this means that φ can be deduced from φ. Indeed, in natural deduction one has a derivation for $\varphi \vdash \varphi$, which consists of a tree of the form

$$[\varphi]^u$$

This derivation means that assuming φ, one concludes φ. In the sequent calculus the simplest rule is the axiom (Ax), that is a sequent with a formula, say φ, that occurs both in the antecedent and in the succedent:

$$\Gamma, \varphi \Rightarrow \varphi, \Delta \quad \text{(Ax)}$$

As a second simple example, consider the sequent

$$\varphi, \varphi \to \psi \Rightarrow \psi$$

This sequent means that ψ is deducible from φ and $\varphi \to \psi$. And in natural deduction one has the corresponding derivation depicted as the tree:

$$\frac{[\varphi]^u \qquad [\varphi \to \psi]^v}{\psi} \ (\to_e)$$

Notice that in the informal interpretation of the sequent $\varphi, \varphi \to \psi \Rightarrow \psi$ it is expressed that the formula ψ in the succedent is derivable from the formulas in the antecedent. Correspondingly, in the natural derivation tree this is expressed by the two undischarged assumptions $[\varphi]^u$ and $[\varphi \to \psi]^v$ and the conclusion ψ.

As we will formally see, the corresponding proof-tree *à la* Gentzen Sequent Calculus is given by the following tree in which the rule $(\mathrm{L}_\to)$, read as "*left* implication", is applied:

$$\frac{\varphi \Rightarrow \varphi \ (\mathrm{Ax}) \qquad \psi \Rightarrow \psi \ (\mathrm{Ax})}{\varphi, \varphi \to \psi \Rightarrow \psi} \ (\mathrm{L}_\to)$$

The intuition with rule $(\mathrm{L}_\to)$ in this deduction is that whenever both φ is deducible from φ and ψ from ψ, ψ is deducible from φ and $\varphi \to \psi$.

From the computational point of view, proofs in a sequent calculus are trees that use more memory in their node labels than proofs in natural deduction. But one has the advantage that in each step of the deductive process all assumptions and conclusions are available directly in the current sequent under consideration, which makes unnecessary searching from assumptions (to be discharged or copied) in previous leaves of the proof-tree.

3.2 A Gentzen's Sequent Calculus for the Predicate Logic

As previously mentioned, sequents are expressions of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *multisets* of formulas. A multiset is a set in which elements can appear repeatedly. Thus, formulas can appear repeatedly in Γ and Δ. The inference rules of the Gentzen sequent calculus for predicate logic are given in the Tables 3.1 and 3.2. The sequent deduction rules are divided into left ("L") and right ("R"), axioms, structural rules and logical rules.

In these rules, Γ and Δ are called the *context* of the rule, the formula in the conclusion of the rule, not in the context, is called the *principal* formula, and the formulas in the premises of the rules, from which the principal formula derives, are called the *active* formulas. In rule (Ax) both occurrences of φ are principal and in $(\mathrm{L}_\perp)$ $\perp$ is principal.

Table 3.1: Axioms and structural rules of Gentzen's SC for predicate logic

left rules	right rules
Axioms:	
$\bot, \Gamma \Rightarrow \Delta$ $(\text{L}_\bot)$	$\Gamma, \varphi \Rightarrow \varphi, \Delta$ (Ax)
Structural rules:	
$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ (L*Weakening*)	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$ (R*Weakening*)
$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ (L*Contraction*)	$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$ (R*Contraction*)

An important observation is that in sequent calculus the syntax does not includes negation ($\neg$). Thus, there are no logical rules for negation in Gentzen's sequent calculus. Negation of a formula φ, that is $\neg\varphi$, would be used here as a shortcut for the formula $\varphi \to \bot$.

The weakening structural rules, for short denoted as (RW) and (LW), mean that whenever Δ holds from Γ, Δ holds from Γ *and* any other formula φ ((LW)) and, from Γ also Δ *or* any other formula φ hold ((RW)). In natural deduction, the intuitive interpretation of weakening rules is that if one has a derivation for $\Gamma \vdash \delta$, also a derivation for $\Gamma, \varphi \vdash \delta$ would be possible ((LW)); on the other side, from $\Gamma \vdash \delta$ one can infer a derivation for $\Gamma \vdash \delta \vee \varphi$ ((RW)). As we will see, some technicalities would be necessary to establish a formal correspondence since in sequent calculus we are working with sequents that are object different to formulas. Indeed, if Δ consists of more than one formula it makes no sense to search for a natural derivation with conclusion Δ.

The contraction structural rules, for short denoted as (RC) and (LC), mean that whenever Δ holds from the set φ, φ, Γ, then Δ still holds if one copy of the duplicated formula φ is deleted from it (case (LC)). On the right side, the analysis of the sequent structural rule (RC) is similar: if the set Δ, φ, φ holds from Γ then Δ, φ, obtained by removing one copy of φ in the succedent, also holds from Γ.

Example 16. *To illustrate the application of the inference rules of Gentzen's sequent calculus*

Table 3.2: Logical rules of Gentzen's sequent calculus for predicate logic

left rules	right rules
Logical rules:	
$\dfrac{\varphi_{i\in\{1,2\}}, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}$ (L$_\wedge$)	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$ (R$_\wedge$)
$\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$ (L$_\vee$)	$\dfrac{\Gamma \Rightarrow \Delta, \varphi_{i\in\{1,2\}}}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}$ (R$_\vee$)
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}$ (L$_\rightarrow$)	$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$ (R$_\rightarrow$)
$\dfrac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall_x \varphi, \Gamma \Rightarrow \Delta}$ (L$_\forall$)	$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/y]}{\Gamma \Rightarrow \Delta, \forall_x \varphi}$ (R$_\forall$), $y \notin \mathtt{fv}(\Gamma, \Delta)$
$\dfrac{\varphi[x/y], \Gamma \Rightarrow \Delta}{\exists_x \varphi, \Gamma \Rightarrow \Delta}$ (L$_\exists$), $y \notin \mathtt{fv}(\Gamma, \Delta)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\Gamma \Rightarrow \Delta, \exists_x \varphi}$ (R$_\exists$)

observe a derivation of Peirce's law below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \;(\mathrm{Ax})}{\varphi \Rightarrow \varphi, \psi}\,(\mathrm{RW})}{\Rightarrow \varphi, \varphi \rightarrow \psi}\,(\mathrm{R}_\rightarrow) \qquad \varphi \Rightarrow \varphi \;(\mathrm{Ax})}{(\varphi \rightarrow \psi) \rightarrow \varphi \Rightarrow \varphi}\,(\mathrm{L}_\rightarrow)}{\Rightarrow ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi}\,(\mathrm{R}_\rightarrow)$$

Observe that the first application of rule (RW) *can be dropped since the sequent* $\varphi \Rightarrow \varphi, \psi$ *is an axiom.*

Example 17. *As a second example consider the following derivation of the sequent* $\varphi \Rightarrow \neg\neg\varphi$*, where* $\neg\varphi$ *is a shortcut for* $\varphi \rightarrow \bot$*, as previously mentioned. Notice that this sequent expresses the natural deduction derived rule* $(\neg\neg_i)$*.*

$$
\cfrac{\cfrac{\cfrac{\varphi \Rightarrow \varphi \;\; (\mathrm{Ax})}{\varphi \Rightarrow \varphi, \bot}\,(\mathrm{RW}) \qquad \cfrac{\bot \Rightarrow \bot \;\; (\mathrm{Ax})}{\varphi, \bot \Rightarrow \bot}\,(\mathrm{LW})}{\varphi \to \bot, \varphi \Rightarrow \bot}\,(\mathrm{L}_{\to})}{\varphi \Rightarrow (\varphi \to \bot) \to \bot}\,(\mathrm{R}_{\to})
$$

As in the previous example, notice that rules (RW) *and* (LW) *are not necessary.*

Example 18. *As a third example consider the following derivation of the sequent* $\neg\neg\varphi \Rightarrow \varphi$*. Notice that this sequent expresses the natural deduction rule* $(\neg\neg_e)$*.*

$$
\cfrac{\cfrac{\varphi \Rightarrow \varphi, \bot \;\; (\mathrm{Ax})}{\Rightarrow \varphi, \varphi \to \bot}\,(\mathrm{R}_{\to}) \qquad \bot \Rightarrow \varphi \;\; (\mathrm{L}_{\bot})}{(\varphi \to \bot) \to \bot \Rightarrow \varphi}\,(\mathrm{L}_{\to})
$$

Exercise 38.

a. *Build a derivation for Modus Tollens; that is, derive the sequent* $\varphi \to \psi, \neg\psi \Rightarrow \neg\varphi$*.*

b. *Build derivations for the contraposition rules,* (CP_1) *and* (CP_2)*; that is, for the sequents* $\varphi \to \psi \Rightarrow \neg\psi \to \neg\varphi$ *and* $\neg\psi \to \neg\varphi \Rightarrow \varphi \to \psi$*.*

c. *Build derivations for the contraposition rules,* (CP_3) *and* (CP_4)*.*

An important observation is that weakening rules are unnecessary. Informally, the possibility of eliminating weakening rules in a derivation is justified by the fact that it would be enough to include the necessary formulas in the context just when *weakened* axioms are allowed, as in our case. When weakening rules are allowed, we only just need *non weakened* axioms of the form "$\varphi \Rightarrow \varphi(\mathrm{Ax})$" and "$\bot \Rightarrow (\mathrm{L}_{\bot})$", which is not the case of our calculus. For instance, observe below a derivation for the sequent $\varphi \Rightarrow \neg\neg\varphi$ without applications of weakening rules.

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi, \bot \ \ (\mathrm{Ax}) \qquad \varphi, \bot \Rightarrow \bot; \ (\mathrm{Ax})}{\varphi \to \bot, \varphi \Rightarrow \bot}\,(\mathrm{L}_{\to})}{\varphi \Rightarrow (\varphi \to \bot) \to \bot}\,(\mathrm{R}_{\to})$$

Exercise 39. (∗) *Prove that weakening rules are unnecessary. It should be proved that all derivations in the sequent calculus can be transformed into a derivation without applications of weakening rules.*

Hint: for doing this you will need to apply induction on the derivations analyzing the case of application of each of the rules just before a last step of weakening. For instance, consider the case of a derivation that finishes in an application of the rule (LW) *after an application of the rule* $(\mathrm{L}_{\to})$*:*

$$\dfrac{\dfrac{\begin{array}{c}\nabla_1\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \Delta\end{array}}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,(\mathrm{L}_{\to})}{\delta, \varphi \to \psi, \Gamma \Rightarrow \Delta}\,(\mathrm{LW})$$

Thus, a new derivation in which rules (LW) *and* $(\mathrm{L}_{\to})$ *are interchanged can be built as below:*

$$\dfrac{\dfrac{\begin{array}{c}\nabla_1\\ \Gamma \Rightarrow \Delta, \varphi\end{array}}{\delta, \Gamma \Rightarrow \Delta, \varphi}\,(\mathrm{LW}) \qquad \dfrac{\begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \Delta\end{array}}{\psi, \delta, \Gamma \Rightarrow \Delta}\,(\mathrm{LW})}{\delta, \varphi \to \psi, \Gamma \Rightarrow \Delta}\,(\mathrm{L}_{\to})$$

Then, by induction hypothesis one can assume the existence of derivations without applications of weakening rules, say ∇'_1 *and* ∇'_2*, for the sequents* $\delta, \Gamma \Rightarrow \Delta, \varphi$ *and* $\psi, \delta, \Gamma \Rightarrow \Delta$*, respectively. Therefore a derivation without application of weakening rules of the form below* *would be possible.*

$$\dfrac{\begin{array}{c}\nabla'_1\\ \delta, \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\nabla'_2\\ \psi, \delta, \Gamma \Rightarrow \Delta\end{array}}{\delta, \varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\mathrm{L}_{\to})$$

An additional detail should be taken in consideration in the application of the induction hypothesis: since other possible applications of weakening rules might appear in the derivations ∇_1 *and* ∇_2*, the correct procedure is starting the elimination of weakening rules from nodes in the proof-tree in which a first application of a weakening rule is done.*

Although the previous rules are sufficient (even dropping the weakening ones) for deduction in the predicate calculus, a useful rule called *cut rule* can be added. Among the applications of the cut rule, its inclusion in the sequent calculus is useful for proving that natural deduction and deduction in the sequent calculus are equivalent.

Table 3.3: Cut Rule

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma\Gamma' \Rightarrow \Delta\Delta'}\ (\mathrm{Cut})$$

In the given rule (Cut), φ is the principal formula, and $\Gamma, \Gamma', \Delta, \Delta'$ is the context. This is a so called *non sharing context* version of (Cut). Also, a so called *sharing context* version of (Cut) is possible in which $\Gamma = \Gamma'$, $\Delta = \Delta'$ and the conclusion is the sequent $\Gamma \Rightarrow \Delta$.

Intuitively, the cut rule allows for inclusion of *lemmas* in proofs: whenever one knows that φ is deducible in a context Γ, Δ and, additionally, one knows that the sequent $\varphi, \Gamma' \Rightarrow \Delta'$ is provable, then one can deduce the conclusion of the cut rule (see the next examples).

Example 19. *To be more illustrative, once a proof for the sequent* $\Rightarrow \neg\neg(\psi \vee \neg\psi)$ *is obtained, the previous proof for the sequent* $\neg\neg\varphi \Rightarrow \varphi$ *(see Example 18) can be used, replacing* φ *by* $\psi \vee \neg\psi$*, to conclude by application of the cut rule that* $\Rightarrow \psi \vee \neg\psi$ *holds:*

$$\frac{\Rightarrow \neg\neg(\psi \vee \neg\psi) \qquad \neg\neg(\psi \vee \neg\psi) \Rightarrow \psi \vee \neg\psi}{\Rightarrow \psi \vee \neg\psi} \text{ (Cut)}$$

Example 20. *Also, a derivation for the sequent* $\Rightarrow \neg\neg(\psi \vee \neg\psi)$ *can be obtained applying the* (Cut) *rule using the previously proved sequent* $\varphi \Rightarrow \neg\neg\varphi$ *(see example 17), replacing* φ *by* $\psi \vee \neg\psi$*, and the sequent* $\Rightarrow \varphi \vee \neg\varphi$*:*

$$\frac{\Rightarrow \psi \vee \neg\psi \qquad \psi \vee \neg\psi \Rightarrow \neg\neg(\psi \vee \neg\psi)}{\Rightarrow \neg\neg(\psi \vee \neg\psi)} \text{ (Cut)}$$

Derivations that do not use the cut rule own an important property called the *subformula* property. Essentially, this property states that the logical rules applied in the derivation can be restricted exclusively to rules for the logical connectives that appear in the sequent in the conclusion of the derivation and, that all formulas that appear in the whole derivation are contained in the conclusion. Indeed, this property is trivially broken when the cut rule is allowed since the principal formula of an application of the cut rule does not need to belong to the conclusion of the derivation. Intuitively, the cut rule enables the use of arbitrary lemmas in the proof of a theorem.

The theorem of *cut elimination* establishes that any proof in the sequent calculus for predicate logic can be transformed in a proof without the use of the cut rule. The proof is elaborated and will not be presented here.

Theorem 12 (Cut Elimination)**.** *Any sequent* $\Gamma \Rightarrow \Delta$ *that is provable with the sequent calculus together with the cut rule is also provable without the latter rule.*

Among the myriad applications of the subterm property and cut elimination theorem, important implications in the structure of proofs can be highlighted, that would be crucial for discriminating between minimal, intuitionistic and classical theorems as we will see in the next section. For instance, they imply the existence of a derivation of the sequent for the law of excluded middle $\Rightarrow \varphi \vee \neg\varphi$ that should use only (axioms and) logical rules for disjunction ($(R_\vee)$ and $(L_\vee)$) and for implication ($(R_\to)$ and $(L_\to)$). Thus, if one applies initially the logical rule $(R_\vee)$ as below, only two non derivable sequents will be obtained: $\Rightarrow \varphi$ and $\Rightarrow \neg\varphi$:

$$\frac{\Rightarrow \varphi}{\Rightarrow \varphi \vee \neg\varphi}\,(R_\vee) \qquad\qquad \frac{\Rightarrow \neg\varphi}{\Rightarrow \varphi \vee \neg\varphi}\,(R_\vee)$$

This implies the necessity of the application of a structural rule before any application of $(R_\vee)$, being the unique option rule (RC):

$$\frac{\Rightarrow \varphi \vee \neg\varphi,\ \ \varphi \vee \neg\varphi}{\Rightarrow \varphi \vee \neg\varphi}\,(RC)$$

Exercise 40.

a. *Complete the derivation of the sequent for LEM:* $\Rightarrow \varphi \vee \neg\varphi$.

b. *Build a derivation for the sequent* $\Rightarrow \neg\neg(\varphi \vee \neg\varphi)$ *using neither rule* (Cut) *nor rule* (RC).

As in natural deduction, we will use notation $\vdash \Gamma \Rightarrow \Delta$ meaning that the sequent $\Gamma \Rightarrow \Delta$ is derivable with Gentzen's sequent calculus. To discriminate we will use subscripts: $\vdash_N$, $\vdash_G$ and $\vdash_{G+cut}$ to denote respectively derivation by natural deduction, deduction *à la* Gentzen and deduction *à la* Gentzen using also the cut rule. Using this notation, the cut elimination theorem can be shortly written as below:

$$\vdash_{G+cut} \Gamma \Rightarrow \Delta \quad \text{iff} \quad \vdash_G \Gamma \Rightarrow \Delta$$

In the remaining of this Chapter for the Gentzen's sequent calculus we will understand the calculus with the cut rule.

3.3 The intuitionistic Gentzen's sequent calculus

As for natural deduction, it is also possible to obtain a restricted set of rules for the intuitionistic logic. It is only necessary to restrict all Gentzen's rules in Tables 3.1 and 3.2 to deal only with sequents with at most one formula in their succedents. For the minimal logic, all sequents in a derivation should have exactly one formula in their succedents. Thus, the rule (RC) should be dropped from the intuitionistic set of Gentzen's rules and, in the intuitionistic case, but not in the minimal one, the rule (RW) might be applied only to sequents with empty succedent:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\ (\mathrm{RW})$$

Essentially, all occurrences of Δ in Tables 3.1 and 3.2 should be adequately replaced by either none or a unique formula, say δ, except for rule (RC) that should be dropped and rule ($\mathrm{L}_{\rightarrow}$) that should be changed into the specialized rule in Table 3.4.

Table 3.4: Left implication rule ($\mathrm{L}_{\rightarrow}$) for the intuitionistic SC

$$\frac{\Gamma \Rightarrow \varphi \qquad \psi, \Gamma \Rightarrow \delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \delta}\ (\mathrm{L}_{\rightarrow})$$

Also, a special version of the cut rule is required as given in Table 3.5.

Table 3.5: Rule (Cut) for the intuitionistic SC

$$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Gamma' \Rightarrow \delta}{\Gamma\Gamma' \Rightarrow \delta}\ (\mathrm{Cut})$$

Example 21. *Observe the derivation below for the sequent* $\Rightarrow \neg\neg\varphi \to \varphi$*, that is related with the non intuitionistic property of elimination of the double negation:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi, \bot \;(\text{Ax})}{\Rightarrow \varphi, \neg\varphi}(\text{R}_\to) \qquad \bot \Rightarrow \varphi, \varphi \;(\text{L}_\bot)}{\neg\neg\varphi \Rightarrow \varphi, \varphi}(\text{L}_\to)}{\neg\neg\varphi \Rightarrow \varphi}(\text{RC})}{\Rightarrow \neg\neg\varphi \to \varphi}(\text{R}_\to)
$$

Since we know that this property is not intuitionistic, there would not be possible derivation of this sequent with the intuitionistic Gentzen's rules; that means, that any possible derivation of this sequent will include a sequent with a succedent with more than one formula (Cf. Example 18).

Observe that the same happens for the sequent $\Rightarrow \varphi \vee \neg\varphi$ *(Cf. Exercise 40).*

Exercise 41. *(Cf. Exercise 40) Build a minimal derivation in the sequent calculus for the sequent* $\Rightarrow \neg\neg(\varphi \vee \neg\varphi)$*.*

Observe that derivations for the sequents for *Modus Tollens* in Exercise 38 can be built in the intuitionistic Gentzen's calculus as well as for the sequent for (CP_1), but not for (CP_2).

Exercise 42 (Cf. Exercise 38)**.** *Give either intuitionistic or classical proofs à la Gentzen for all Gentzen's versions of* (CP) *according to your answers to Exercises 9 and 10.*

a. $\varphi \to \psi \Rightarrow \neg\psi \to \neg\varphi$ (CP_1)*;*

b. $\neg\varphi \to \neg\psi \Rightarrow \psi \to \varphi$ (CP_2)*;*

c. $\neg\varphi \to \psi \Rightarrow \neg\psi \to \varphi$ (CP_3)*; and*

d. $\varphi \to \neg\psi \Rightarrow \psi \to \neg\varphi$ (CP_4)*.*

Exercise 43 (Cf. Exercise 38)**.** *Also, provide intuitionistic or classical derivations for the versions below of Modus Tollens, according to your classification in Exercise 11.*

a. $\varphi \to \psi, \neg\psi \Rightarrow \neg\varphi$ (MT_1)*;*

b. $\varphi \to \neg\psi, \psi \Rightarrow \neg\varphi$ (MT_2)*;*

c. $\neg\varphi \to \psi, \neg\psi \Rightarrow \varphi$ (MT_3)*; and*

d. $\neg\varphi \to \neg\psi, \psi \Rightarrow \neg\varphi$ (MT_4).

Example 22 (Cf. Example 18)**.** *Consider the following classical derivation of the sequent* $\Rightarrow \forall_x(\neg\neg\varphi \to \varphi)$.

$$\dfrac{\dfrac{\dfrac{(\mathrm{R}_{\to})\,\dfrac{\varphi \Rightarrow \varphi, \bot\ \ (\mathrm{Ax})}{\Rightarrow \varphi, \neg\varphi} \qquad\qquad \bot \Rightarrow \varphi\ \ (\mathrm{L}_{\bot})}{\neg\neg\varphi \Rightarrow \varphi}\,(\mathrm{L}_{\to})}{\Rightarrow \neg\neg\varphi \to \varphi}\,(\mathrm{R}_{\to})}{\Rightarrow \forall_x(\neg\neg\varphi \to \varphi)}\,(\mathrm{R}_{\forall})$$

Sequents of the form $\Rightarrow \forall(\neg\neg\varphi \to \varphi)$ are called *stability axioms* and are derivable in the strict classical calculus. There is no possible intuitionistic derivation for this kind of sequent. In fact, the reader can notice that this is related with the strictly classical rule $(\neg\neg_e)$ in deduction natural. Also, the reader can check that the use of the classical rule $(\mathrm{L}_{\to})$ as well as the inclusion of sequents with more than one formula in the succedent are obligatory to build a derivation for this kind of sequents.

Exercise 44.

1. *Build an intuitionistic derivation for the sequent* $\Rightarrow \neg\neg(\neg\neg\varphi \to \varphi)$.
2. *Build a non classical derivation for the double negation of Peirce's law:* $\Rightarrow \neg\neg(((\varphi \to \psi) \to \varphi) \to \varphi)$.

Exercise 45 (Cf. Exercise 12)**.** *Using the intuitionistic Gentzen's calculus build derivations for the following sequents.*

a. $\neg\neg\neg\phi \Rightarrow \neg\phi$ *and* $\neg\phi \Rightarrow \neg\neg\neg\phi$

b. $\neg\neg(\phi \to \psi) \Rightarrow (\neg\neg\phi \to \neg\neg\psi)$.

c. $\neg\neg(\phi \land \psi) \Rightarrow (\neg\neg\phi \land \neg\neg\psi)$.

d. $\neg(\phi \lor \psi) \Rightarrow (\neg\phi \land \neg\psi)$ *and* $(\neg\phi \land \neg\psi) \Rightarrow \neg(\phi \lor \psi)$.

3.4 Natural Deduction versus Deduction *à la* Gentzen

In this section we prove that both natural deduction and deduction *à la* Gentzen have the same expressive power, that means that we can prove exactly the same set of theorems using natural deduction or using deduction *à la* Gentzen. Initially, we prove that the property holds restricted to the intuitionistic logic. Then, we prove that it holds for the logic of predicates.

The main result is stated as

$$\vdash_G \Gamma \Rightarrow \varphi \quad \text{if and only if} \quad \Gamma \vdash_N \varphi$$

For proving this result, we will use an informal style of discussion which requires a deal of additional effort of the reader in order to interpret a few points that would not be presented in detail. Among others points, notice for instance that the antecedent "Γ" of the sequent $\Gamma \Rightarrow \varphi$ is in fact a *multiset* of formulas, while "Γ" as premise of $\Gamma \vdash_N \varphi$ should be interpreted as a *finite subset* of assumptions built from Γ that can be used in a natural derivation of φ.

Notice also, that in the classical sequent calculus one can build derivations for sequents of the form $\Gamma \Rightarrow \Delta$, and in natural deduction only derivations of a formula, say δ, are allowed, that is derivations of the form $\Gamma' \vdash_N \delta$. Then for the classical logic it would be necessary to establish a correspondence between derivability of arbitrary sequents of the form $\Gamma \Rightarrow \Delta$ and derivability of "equivalent" sequents with exactly one formula in the succedent of the form $\Gamma' \Rightarrow \delta$.

3.4.1 Equivalence between ND and Gentzen's SC - the intuitionistic case

The equivalence for the case of the intuitionistic logic is established in the next theorem.

Theorem 13 (ND versus SC for the intuitionistic logic). *The equivalence below holds for the intuitionistic sequent calculus and the intuitionistic natural deduction:*

$$\vdash_G \Gamma \Rightarrow \varphi \text{ if and only if } \Gamma \vdash_N \varphi$$

Proof. According to previous observations, it is possible to consider the calculus *à la* Gentzen without weakening rules. We will prove that the intuitionistic Gentzen's sequent calculus, including the cut rule, is equivalent to intuitionistic natural deduction. The proof is by induction on the structure of derivations.

Initially, we prove necessity, that is $\vdash_G \Gamma \Rightarrow \varphi$ implies $\Gamma \vdash_N \varphi$. This is done by induction on derivations in the intuitionistic Gentzen's sequent calculus, analyzing different cases according to the last rule applied in a derivation.

IB. The simplest derivations *à la* Gentzen are given by applications of rules (Ax) and ($L_\perp$):

$\Gamma, \varphi \Rightarrow \varphi$ (Ax) $\qquad \Gamma, \perp \Rightarrow \varphi$ ($L_\perp$)

In natural deduction, these proofs correspond respectively to derivations:

$$[\varphi]^u \text{ (Ax)} \qquad\qquad \frac{[\perp]^u}{\varphi}\ (\perp_e)$$

Notice that this means that $\Gamma, \varphi \vdash_N \varphi$ and $\Gamma, \perp \vdash_N \varphi$, since the assumption of the former derivation φ belongs to $\Gamma \cup \{\varphi\}$ and the assumption of the latter derivation $\perp$ belongs to $\Gamma \cup \{\perp\}$.

IS. We will consider derivations in the Gentzen calculus analyzing cases according to the last rule applied in the derivation. Right rules correspond to introduction rules, and left rules will need a more elaborated analysis. First, observe that in the intuitionistic case the sole contraction rule to be considered is (LC):

$$\dfrac{\begin{array}{c}\nabla\\ \psi,\psi,\Gamma\Rightarrow\varphi\end{array}}{\psi,\Gamma\Rightarrow\varphi}\ (\mathrm{LC})$$

And, whenever we have a derivation finishing in an application of this rule, by induction hypothesis, there is a natural derivation of its premise $\{\psi\}\cup\{\psi\}\cup\Gamma\vdash_N\varphi$, which corresponds to $\{\psi\}\cup\Gamma\vdash_N\varphi$ because the premises in natural deduction are sets.

Case ($\mathrm{L}_\wedge$). Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \psi,\Gamma\Rightarrow\varphi\end{array}}{\psi\wedge\delta,\Gamma\Rightarrow\varphi}\ (\mathrm{L}_\wedge)$$

By induction hypothesis, one has a derivation for $\Gamma,\psi\vdash_N\varphi$, say ∇', whose assumptions are ψ and a finite subset Γ' of Γ. Thus a natural derivation is obtained as below, by replacing each occurrence of the assumption $[\psi]$ in ∇' by an application of rule $(\wedge_e)$.

$$\begin{array}{c}\dfrac{[\psi\wedge\delta]^u}{\psi}\ (\wedge_e)\\ \nabla'\\ \varphi\end{array}$$

By brevity, in the previous derivation assumptions in Γ' were dropped, as will be done in all other derivations in this proof.

Case ($\mathrm{R}_\wedge$). Suppose $\varphi=\delta\wedge\psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla_1\\ \Gamma\Rightarrow\delta\end{array}\qquad\begin{array}{c}\nabla_2\\ \Gamma\Rightarrow\psi\end{array}}{\Gamma\Rightarrow\delta\wedge\psi}\ (\mathrm{R}_\wedge)$$

By induction hypothesis, one has derivations for $\Gamma\vdash_N\delta$ and $\Gamma\vdash_N\psi$, say ∇'_1 and ∇'_2. Thus, a natural derivation is built from these derivations applying the rule $(\wedge_i)$ as below.

$$\dfrac{\begin{array}{c}\nabla_1'\\ \delta\end{array} \qquad \begin{array}{c}\nabla_2'\\ \psi\end{array}}{\delta \wedge \psi}\ (\wedge_i)$$

Case ($\mathrm{L}_\vee$). Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla_1\\ \delta, \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\delta \vee \psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\vee)$$

By induction hypothesis, one has derivations ∇_1' and ∇_2' for $\delta, \Gamma \vdash_N \varphi$ and $\psi, \Gamma \vdash_N \varphi$. Thus, a natural derivation, that assumes $\delta \vee \psi$, is obtained from these derivations applying the rule $(\vee_e)$ as below.

$$\dfrac{[\delta \vee \psi]^u \qquad \begin{array}{c}[\delta]^v\\ \nabla_1'\\ \varphi\end{array} \qquad \begin{array}{c}[\psi]^w\\ \nabla_2'\\ \varphi\end{array}}{\varphi}\ (\vee_e)\ v, w$$

Case ($\mathrm{R}_\vee$). Suppose $\varphi = \delta \vee \psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \Gamma \Rightarrow \delta\end{array}}{\Gamma \Rightarrow \delta \vee \psi}\ (\mathrm{R}_\vee)$$

By induction hypotheses there exists a natural derivation ∇' for $\Gamma \vdash_N \delta$. Applying at the end of this derivation rule $(\vee_i)$, one obtains a natural derivation for $\Gamma \vdash_N \delta \vee \psi$.

Case ($\mathrm{L}_\rightarrow$). Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla_1\\ \Gamma \Rightarrow \delta\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\delta \rightarrow \psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\rightarrow)$$

By induction hypothesis there exist natural derivations ∇'_1 and ∇'_2 for $\Gamma \vdash_N \delta$ and $\psi, \Gamma \vdash_N \varphi$. A natural derivation for $\Gamma \vdash_N \varphi$ is obtained from these derivations, by replacing each assumption $[\psi]^u$ in ∇'_2 by a derivation of ψ finishing in an application of rule $(\rightarrow_e)$ with premises $[\delta \rightarrow \psi]^v$ and δ. The former as a new assumption and the latter is derived as in ∇'_1.

$$\dfrac{[\delta \rightarrow \psi]^v \qquad \begin{array}{c}\nabla'_1\\ \delta\end{array}}{\begin{array}{c}\psi\\ \nabla'_2\\ \varphi\end{array}}\ (\rightarrow_e)$$

Case $(\mathrm{R}_{\rightarrow})$. Suppose $\varphi = \delta \rightarrow \psi$ and one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \delta, \Gamma \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \delta \rightarrow \psi}\ (\mathrm{R}_{\rightarrow})$$

By induction hypothesis, there exists a natural derivation ∇' for $\delta, \Gamma \vdash_N \psi$. The natural derivation for $\Gamma \vdash_N \delta \rightarrow \psi$ is obtained by applying at the end of this proof rule $(\rightarrow_i)$ discharging assumptions $[\delta]^u$ as depicted below.

$$\dfrac{\begin{array}{c}[\delta]^u\\ \nabla'\\ \psi\end{array}}{\delta \rightarrow \psi}\ (\rightarrow_i)\ u$$

Case $(\mathrm{L}_{\forall})$. Suppose one has a derivation of the form

$$\dfrac{\begin{array}{c}\nabla\\ \psi[x/y], \Gamma \Rightarrow \varphi\end{array}}{\forall_x \psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_{\forall})$$

Then by induction hypothesis there exists a natural derivation for $\psi[x/y], \Gamma \vdash_N \varphi$, say ∇'.

A natural derivation for $\forall_x\psi, \Gamma \vdash_N \varphi$ is obtained by replacing all assumptions of $[\psi[x/y]]^u$ in ∇' by a deduction of $\psi[x/y]$ with assumption $[\forall_x\psi]^v$ applying rule $(\forall_e)$.

Case $(\mathrm{R}_\forall)$. Suppose $\varphi = \forall_x\psi$ and one has a derivation of the form

$$\dfrac{\overset{\nabla}{\Gamma \Rightarrow \psi[x/y]}}{\Gamma \Rightarrow \forall_x\psi}\ (\mathrm{R}_\forall)$$

where $y \notin \mathtt{fv}(\Gamma)$. Then by induction hypothesis there exists a natural derivation ∇' for $\Gamma \vdash_N \psi[x/y]$. Thus a simple application at the end of ∇' of rule $(\forall_i)$, that is possible since y does not appears in the open assumptions, will complete the desired natural derivation.

Case $(\mathrm{L}_\exists)$. Suppose one has a derivation of the form

$$\dfrac{\overset{\nabla}{\psi[x/y], \Gamma \Rightarrow \varphi}}{\exists_x\psi, \Gamma \Rightarrow \varphi}\ (\mathrm{L}_\exists)$$

where $y \notin \mathtt{fv}(\Gamma, \varphi)$. By induction hypothesis there exists a natural derivation ∇' for $\psi[x/y], \Gamma \vdash_N \varphi$. The desired derivation is built by an application of rule $(\exists_e)$ using as premises the assumption $[\exists_x\psi]^v$ and the conclusion of ∇'. In this application assumptions of $[\psi[x/y]]^u$ in ∇' are discharged as depicted below. Notice that the application of rule $(\exists_e)$ is possible since $y \notin \mathtt{fv}(\Gamma, \varphi)$, which implies it does not will appear in open assumptions in ∇'.

$$\dfrac{[\exists_x\psi]^v \qquad \begin{array}{c}[\psi[x/y]]^u\\ \nabla'\\ \varphi\end{array}}{\varphi}\ (\exists_e)\ u$$

Case $(\mathrm{R}_\exists)$. Suppose $\varphi = \exists_x\psi$ and one has a derivation of the form

$$\dfrac{\overset{\nabla}{\Gamma \Rightarrow \psi[x/t]}}{\Gamma \Rightarrow \exists_x\psi}\ (\mathrm{R}_\exists)$$

A natural derivation for $\Gamma \vdash_N \exists_x \psi$ is built by induction hypothesis which gives a natural derivation ∇' for $\Gamma \vdash_N \psi[x/t]$ and application of rule $(\exists_i)$ to the conclusion of ∇'.

Case (cut). Suppose one has a derivation finishing in an application of rule (Cut) as below

$$\dfrac{\begin{array}{c}\nabla_1\\ \Gamma \Rightarrow \psi\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\Gamma \Rightarrow \varphi}\ (\text{Cut})$$

By induction hypothesis there are natural derivations ∇'_1 and ∇'_2 for $\Gamma \vdash_N \psi$ and $\psi, \Gamma \vdash_N \varphi$. To obtain the desired natural derivation, all assumptions $[\psi]^u$ in ∇'_2 are replaced by derivations of ψ using ∇'_1:

$$\begin{array}{c}\nabla'_1\\ \psi\\ \nabla'_2\\ \varphi\end{array}$$

Now we prove sufficiency, that is $\vdash_G \Gamma \Rightarrow \varphi$ whenever $\Gamma \vdash_N \varphi$. The proof is by induction on the structure of natural derivations analyzing the last applied rule.

IB. Proofs consisting of a sole node $[\varphi]^u$ correspond to applications of (Ax): $\Gamma \Rightarrow \varphi$, where $\varphi \in \Gamma$.

IS. All derivations finishing in introduction rules are straightforwardly related with derivations *à la* Gentzen finishing in the corresponding right rule as in the proof of necessity. Only one example is given: $(\rightarrow_i)$. The other cases are left as an exercise for the reader.

Suppose $\varphi = \delta \rightarrow \psi$ and one has a derivation finishing in an application of $(\rightarrow_i)$ discharging assumptions of δ and using assumptions in Γ:

$$\dfrac{\begin{array}{c}[\delta]^u\\ \nabla\\ \psi\end{array}}{\delta \rightarrow \psi}\ (\rightarrow_i)\ u$$

By induction hypothesis there exists a derivation *à la* Gentzen ∇' for the sequent $\delta, \Gamma \Rightarrow \psi$.

Thus, the desired derivation is built by a simple application of rule ($R_\rightarrow$):

$$\dfrac{\begin{array}{c}\nabla' \\ \delta, \Gamma \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \delta \rightarrow \psi}\ (R_\rightarrow)$$

Derivations finishing in elimination rules will require application of the rule (Cut). A few interesting cases are given. All the other cases remain as an exercise for the reader.

Case ($\vee_e$). Suppose one has a natural derivation for $\Gamma \vdash_N \varphi$ finishing in an application of rule ($\vee_e$) as below.

$$\dfrac{\begin{array}{c}\nabla \\ \delta \vee \psi\end{array} \qquad \begin{array}{c}[\delta]^v \\ \nabla_1 \\ \varphi\end{array} \qquad \begin{array}{c}[\psi]^w \\ \nabla_2 \\ \varphi\end{array}}{\varphi}\ (\vee_e)\ v, w$$

By induction hypothesis, there are derivations *à la* Gentzen ∇', ∇'_1 and ∇'_2 respectively, for the sequents $\Gamma \Rightarrow \delta \vee \psi$, $\delta, \Gamma \Rightarrow \varphi$ and $\psi, \Gamma \Rightarrow \varphi$. Thus, using these derivations, a derivation for $\Gamma \Rightarrow \varphi$ is built as below.

$$\dfrac{\begin{array}{c}\nabla' \\ \Gamma \Rightarrow \delta \vee \psi\end{array} \qquad \dfrac{\begin{array}{c}\nabla'_1 \\ \delta, \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\nabla'_2 \\ \psi, \Gamma \Rightarrow \varphi\end{array}}{\delta \vee \psi, \Gamma \Rightarrow \varphi}\ (L_\vee)}{\Gamma \Rightarrow \varphi}\ (\text{Cut})$$

Case ($\rightarrow_e$). Suppose one has a natural derivation for $\Gamma \vdash_N \varphi$ that finishes in an application of ($\rightarrow_e$) as below.

$$\dfrac{\begin{array}{c}\nabla_1 \\ \delta\end{array} \qquad \begin{array}{c}\nabla_2 \\ \delta \rightarrow \varphi\end{array}}{\varphi}\ (\rightarrow_e)$$

By induction hypothesis, there are derivations *à la* Gentzen ∇'_1 and ∇'_2 for the sequents

$\Gamma \Rightarrow \delta$ and $\Gamma \Rightarrow \delta \to \varphi$, respectively. The desired derivation is built, using these derivations, as below.

$$\dfrac{\begin{array}{c}\nabla_2' \\ \Gamma \Rightarrow \delta \to \varphi\end{array} \qquad \dfrac{\begin{array}{c}\nabla_1' \\ \Gamma \Rightarrow \delta\end{array} \qquad \varphi, \Gamma \Rightarrow \varphi\ (\mathrm{Ax})}{\delta \to \varphi, \Gamma \Rightarrow \varphi}\,(\mathrm{L}_{\to})}{\Gamma \Rightarrow \varphi}\,(\mathrm{Cut})$$

Case $(\exists_e)$. Suppose one has a natural derivation for $\Gamma \vdash_N \varphi$ finishing in an application of the rule $(\exists_e)$ as below.

$$\dfrac{\begin{array}{c}\nabla_1 \\ [\exists_x \psi]^v\end{array} \qquad \begin{array}{c}[\psi[x/y]]^u \\ \nabla_2 \\ \varphi\end{array}}{\varphi}\,(\exists_e)\ u$$

By induction hypothesis, there are derivations *à la* Gentzen ∇_1' and ∇_2' for the sequents $\Gamma \Rightarrow \exists_x \psi$ and $\psi[x/y], \Gamma \Rightarrow \varphi$, respectively. The derivation is built as below. Notice that $y \notin \mathtt{fv}(\Gamma, \varphi)$, which allows the application of the rule $(\mathrm{L}_\exists)$.

$$\dfrac{\begin{array}{c}\nabla_1' \\ \Gamma \Rightarrow \exists_x \psi\end{array} \qquad \dfrac{\begin{array}{c}\nabla_2' \\ \psi[x/y], \Gamma \Rightarrow \varphi\end{array}}{\exists_x \psi, \Gamma \Rightarrow \varphi}\,(\mathrm{L}_\exists)}{\Gamma \Rightarrow \varphi}\,(\mathrm{Cut})$$

□

Exercise 46. *Prove all remaining cases in the proof of sufficiency of Theorem 13.*

3.4.2 Equivalence of ND and Gentzen's SC - the classical case

Before proving equivalence of natural deduction and deduction *à la* Gentzen for predicate logic, a few additional definitions and properties are necessary. First of all, we define a notion that makes it possible to transform any sequent in an equivalent one but with only one formula in its succedent.

By $\overline{\Gamma}$ we generically denote any sequence of formulas built from the formulas in the sequence Γ, replacing each formula in Γ by either its negation or, when the head symbol of the formula is the negation symbol, eliminating it from the formula. For instance, let $\Delta = \delta_1, \neg\delta_2, \neg\delta_3, \delta_4$, then $\overline{\Delta}$ might represent sequences as $\neg\delta_1, \neg\neg\delta_2, \delta_3, \neg\delta_4$; $\neg\delta_1, \delta_2, \delta_3, \neg\delta_4$, etc. This transformation is not only relevant for our purposes in this chapter, but also in computational frameworks, as we will see in the next chapter, in order to get rid automatically of negative formulas in sequents that appear in a derivation.

Definition 30 (c-equivalent sequents). *We will say that sequents $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \neg\varphi$ as well as $\Gamma \Rightarrow \Delta, \varphi$ and $\neg\varphi, \Gamma \Rightarrow \Delta$ are c-equivalent in one-step. The equivalence closure of this relation is called the c-equivalence relation on sequents and is denoted as $\equiv_{ce}$.*

According to the previous notational convention, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ and $\Gamma, \overline{\Delta'} \Rightarrow \Delta, \overline{\Gamma'}$ are c-equivalent; that is,

$$\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad \equiv_{ce} \quad \Gamma, \overline{\Delta'} \Rightarrow \Delta, \overline{\Gamma'}$$

Lemma 8 (One-step c-equivalence). *The following properties hold in the sequent calculus à la Gentzen for the classical logic:*

i) There exists a derivation for $\vdash_G \varphi, \Gamma \Rightarrow \Delta$, if and only if there exists a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \neg\varphi$.

ii) There is a derivation for $\vdash_G \neg\varphi, \Gamma \Rightarrow \Delta$, if and only if there is a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \varphi$.

Proof. We consider the derivations below.

i) **Necessity**: Let ∇ be a derivation for $\vdash_G \varphi, \Gamma \Rightarrow \Delta$. Then the desired derivation is built as follows:

$$\dfrac{\dfrac{\begin{array}{c}\nabla\\ \varphi, \Gamma \Rightarrow \Delta\end{array}}{\varphi, \Gamma \Rightarrow \Delta, \bot}\,(\mathrm{RW})}{\Gamma \Rightarrow \Delta, \neg\varphi}\,(\mathrm{R}_{\rightarrow})$$

Sufficiency: Let ∇ be a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \neg\varphi$. Then the desired derivation is built as follows:

$$\dfrac{(\mathrm{LW})\ \dfrac{\overset{\nabla}{\Gamma \Rightarrow \Delta, \neg\varphi}}{\varphi, \Gamma \Rightarrow \Delta, \neg\varphi} \qquad \dfrac{(\mathrm{Ax})\ \varphi, \Gamma \Rightarrow \Delta, \varphi \qquad \bot, \varphi, \Gamma \Rightarrow \Delta\ (\mathrm{L}_\bot)}{\neg\varphi, \varphi, \Gamma \Rightarrow \Delta}\ (\mathrm{L}_\to)}{\varphi, \Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

Observe that in both cases, when Δ is the empty sequence we have an intuitionistic proof.

ii) **Necessity**: Let ∇ be a derivation for $\vdash_G \neg\varphi, \Gamma \Rightarrow \Delta$. Then the desired derivation is built as follows:

$$\dfrac{\overset{\nabla'}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi \to \varphi} \qquad \dfrac{\dfrac{\dfrac{\overset{\nabla}{\neg\varphi, \Gamma \Rightarrow \Delta}}{\neg\varphi, \Gamma \Rightarrow \Delta, \varphi, \bot}\ (\mathrm{RW})}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi}\ (\mathrm{R}_\to) \qquad \varphi, \Gamma \Rightarrow \Delta, \varphi\ (\mathrm{Ax})}{\neg\neg\varphi \to \varphi, \Gamma \Rightarrow \Delta, \varphi}\ (\mathrm{L}_\to)}{\Gamma \Rightarrow \Delta, \varphi}\ (\mathrm{Cut})$$

where ∇' is the derivation below:

$$\dfrac{\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \Delta, \varphi, \varphi, \bot\ \ (\mathrm{Ax})}{\Gamma \Rightarrow \Delta, \varphi, \varphi, \neg\varphi}\ (\mathrm{R}_\to) \qquad \bot, \Gamma \Rightarrow \Delta, \varphi, \varphi\ (\mathrm{L}_\bot)}{\neg\neg\varphi, \Gamma \Rightarrow \Delta, \varphi, \varphi}\ (\mathrm{L}_\to)}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi \to \varphi}\ (\mathrm{R}_\to)$$

Observe that this case is strictly classic because the left premise of (Cut), that is the derivation ∇', is essentially a proof of the sequent $\Rightarrow \neg\neg\varphi \to \varphi$ (Also, see examples 18 and 22).

Sufficiency: let ∇ be a derivation for $\vdash_G \Gamma \Rightarrow \Delta, \varphi$. Then the desired derivation is built as follows:

$$\dfrac{\overset{\nabla}{\Gamma \Rightarrow \Delta, \varphi} \qquad \bot, \Gamma \Rightarrow \Delta \;\; (\mathrm{L}_\bot)}{\neg\varphi, \Gamma \Rightarrow \Delta} \; (\mathrm{L}_\to)$$

Observe that in this case, when Δ is the empty sequence we have an intuitionistic proof.

□

Corollary 3 (One-step c-equivalence in the intuitionistic calculus). *The following properties hold in the* **intuitionistic** *calculus à la Gentzen:*

i) There is a derivation for $\vdash_G \varphi, \Gamma \Rightarrow$*, if and only if there is a derivation for* $\vdash_G \Gamma \Rightarrow \neg\varphi$.

ii) Assuming that $\Rightarrow \neg\neg\varphi \to \varphi$*, the existence of a derivation for* $\vdash_G \neg\varphi, \Gamma \Rightarrow$*, implies the existence of a derivation for* $\vdash_G \Gamma \Rightarrow \varphi$.

iii) There exist a derivation for $\vdash_G \neg\varphi, \Gamma \Rightarrow$*, whenever there is a derivation for* $\vdash_G \Gamma \Rightarrow \varphi$.

Proof. The proof is obtained from the proof of Lemma 8, according to the observations given in that proof. In particular for the item ii), the proof of sufficiency of the lemma is easily modified as below.

$$\dfrac{\dfrac{\text{(Assumption)}\quad \Rightarrow \neg\neg\varphi \to \varphi}{\Gamma \Rightarrow \neg\neg\varphi\to\varphi}\,(\mathrm{RW}) \qquad \dfrac{\dfrac{\dfrac{\neg\varphi,\Gamma\Rightarrow}{\neg\varphi,\Gamma\Rightarrow\bot}\,(\mathrm{RW})}{\Gamma\Rightarrow\neg\neg\varphi}\,(\mathrm{R}_\to) \qquad \varphi,\Gamma\Rightarrow\varphi\;(\mathrm{Ax})}{\neg\neg\varphi\to\varphi,\Gamma\Rightarrow\varphi}\,(\mathrm{L}_\to)}{\Gamma\Rightarrow\varphi}\,(\mathrm{Cut})$$

□

Exercise 47. *Complete the proof of the Corollary 3.*

Lemma 9 (c-equivalence)**.** *Let* $\Gamma \Rightarrow \Delta$ *and* $\Gamma' \Rightarrow \Delta'$ *be c-equivalent sequents, that is* $\Gamma \Rightarrow \Delta \equiv_{ce} \Gamma' \Rightarrow \Delta'$*. Then the following holds in the* **classical** *Gentzen's sequent calculus:*

$$\vdash_G \Gamma \Rightarrow \Delta \textit{ if and only if } \vdash_G \Gamma' \Rightarrow \Delta'$$

Proof. (Sketch) Suppose, $\Gamma \Rightarrow \Delta$ equals $\Gamma^1, \Gamma^2 \Rightarrow \Delta^1, \Delta^2$ and $\Gamma' \Rightarrow \Delta'$ equals $\Gamma^1, \overline{\Delta^2} \Rightarrow \Delta^1, \overline{\Gamma^2}$. The proof is by induction on $n = |\Gamma^2, \Delta^2|$, that is the number of switched formulas (from the succedent to the antecedent and vice versa), that are necessary to obtain $\Gamma' \Rightarrow \Delta'$ from $\Gamma \Rightarrow \Delta$ by a number n of one-step c-equivalence transformations. Suppose $\Gamma^1, \Gamma^2_k, \overline{\Delta^2_k} \Rightarrow \Delta^1, \Delta^2_k, \overline{\Gamma^2_k}$, for $0 \leq k \leq n$, is the sequent after k one-step c-equivalence transformations, being $\Gamma^2_0 = \Gamma^2$, $\Delta^2_0 = \Delta^2$ (thus, being $\overline{\Delta^2_0}$ and $\overline{\Gamma^2_0}$ empty sequences) and Γ^2_n and Δ^2_n empty sequences (thus, being $\overline{\Gamma^2_n} = \overline{\Gamma^2}$ and $\overline{\Delta^2_n} = \overline{\Delta^2}$).

In the inductive step, for $k < n$, one assumes that there is a proof of the sequent: $\vdash_G \Gamma^1, \Gamma^2_k, \overline{\Delta^2_k} \Rightarrow \Delta^1, \Delta^2_k, \overline{\Gamma^2_k}$. Thus, applying an one-step c-equivalence transformation, by Lemma 8, one obtains a proof for $\vdash_G \Gamma^1, \Gamma^2_{k+1}, \overline{\Delta^2_{k+1}} \Rightarrow \Delta^1, \Delta^2_{k+1}, \overline{\Gamma^2_{k+1}}$. □

Exercise 48. *Complete all details of the proof of Lemma 9.*

In order to extend the c-equivalence Lemma from classical to intuitionistic logic, it is necessary to assume all necessary stability axioms (Cf. item 2 of Corollary 3).

Definition 31 (Intuitionistic derivability modulo stability axioms)**.** *A stability axiom is a sequent of the form* $\Rightarrow \forall_x(\neg\neg\varphi \to \varphi)$*. Intuitionistic derivability modulo stability axioms is defined as intuitionistic derivability assuming all possible stability axioms. Intuitionistic derivability à la Gentzen with stability axioms will be denoted as* $\vdash_{Gi+St}$*.*

Lemma 10 (Equivalence between classical and intuitionistic SC modulo stability axioms)**.** *For all sequents* $\Gamma \Rightarrow \delta$ *the following property holds:*

$$\vdash_G \Gamma \Rightarrow \delta \textit{ iff } \vdash_{Gi+St} \Gamma \Rightarrow \delta$$

Therefore, for any sequent $\Gamma' \Rightarrow \Delta'$ *c-equivalent to* $\Gamma \Rightarrow \delta$*,* $\vdash_G \Gamma' \Rightarrow \Delta'$ *iff* $\vdash_{Gi+St} \Gamma \Rightarrow \delta$*.*

Proof. (Sketch) To prove that $\vdash_{Gi+St} \Gamma \Rightarrow \delta$ implies $\vdash_G \Gamma \Rightarrow \delta$, suppose that ∇ is a derivation for $\vdash_{Gi+St} \Gamma \Rightarrow \delta$. The derivation ∇ is transformed into a classical derivation in the following manner: for any stability axiom assumption, that is a sequent of the form $\Rightarrow \forall_x(\neg\neg\varphi \to \varphi)$ that appears as a leaf in the derivation ∇, replace the assumption by a classical proof for $\vdash_G \ \Rightarrow \forall_x(\neg\neg\varphi \to \varphi)$. In this way, after all stability axiom assumptions are replaced by classical derivations, one obtains a classical derivation, say ∇', for $\vdash_G \Gamma \Rightarrow \delta$. Additionally, by Lemma 9, $\vdash_G \Gamma \Rightarrow \delta$ if and only if there exists a classical derivation for $\vdash_G \Gamma' \Rightarrow \Delta'$.

To prove that $\vdash_{Gi+St} \Gamma \Rightarrow \delta$ whenever $\vdash_G \Gamma \Rightarrow \delta$, one applies induction on the structure of the classical derivation. Most rules require a direct analysis, for instance the inductive step for rule $(R_\to)$ is given below.

Case $(R_\to)$. The derivation is of the form given below.

$$\dfrac{\begin{array}{c}\nabla\\ \Gamma, \varphi \Rightarrow \psi\end{array}}{\Gamma \Rightarrow' \varphi \to \psi}\ (R_\to)$$

By induction hypothesis there exists a derivation ∇' for $\vdash_{Gi+St} \Gamma, \varphi \Rightarrow \psi$, Thus, the desired derivation is obtained simply by an additional application of rule $(R_\to)$ to the conclusion of the intuitionistic derivation ∇'.

The interesting case happens for rule $(L_\rightarrow)$ since this rule requires two formulas in the succedent of one of the premises. The analysis of the inductive step for rule $(L_\rightarrow)$ is given below.

Case $(L_\rightarrow)$. The last step of the proof is of the form below, where $\Gamma = \Gamma'', \varphi \rightarrow \psi$.

$$\dfrac{\begin{array}{c}\nabla_1\\ \Gamma'' \Rightarrow \delta, \varphi\end{array} \qquad \begin{array}{c}\nabla_2\\ \psi, \Gamma'' \Rightarrow \delta\end{array}}{\Gamma'', \varphi \rightarrow \psi \Rightarrow \delta}\ (L_\rightarrow)$$

By induction hypothesis there exist derivations, say ∇_1' and ∇_2', for $\vdash_{Gi+St} \Gamma'', \neg\delta \Rightarrow \varphi$ and $\vdash_{Gi+St} \psi, \Gamma, \neg\delta \Rightarrow$. Notice that the argumentation is not as straightforwardly as it appears, since it is necessary to build first classical derivations for $\vdash_G \Gamma'', \neg\delta \Rightarrow \varphi$ and $\vdash_G \psi, \Gamma'', \neg\delta \Rightarrow$ using (Lemma 8 and) Corollary 3.

Thus, a derivation for $\vdash_{Gi+St} \Gamma'', \varphi \rightarrow \psi, \neg\delta \Rightarrow$ is obtained as below.

$$\dfrac{\begin{array}{c}\nabla_1'\\ \Gamma'', \neg\delta \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\nabla_2'\\ \psi, \Gamma'', \neg\delta \Rightarrow\end{array}}{\Gamma'', \varphi \rightarrow \psi, \neg\delta \Rightarrow}\ (L_\rightarrow)$$

By a final application of Corollary 3 there exists a derivation for $\vdash_{Gi+St} \Gamma, \varphi \rightarrow \psi \Rightarrow \delta$. □

Exercise 49. *Prove the remaining cases of the proof of Lemma 10.*

Theorem 14 (Natural versus deduction *à la* Gentzen for the classical logic)**.** *One has that for the classical Gentzen and natural calculus*

$$\vdash_G \Gamma \Rightarrow \varphi \textit{ if and only if } \Gamma \vdash_N \varphi$$

Proof. (Sketch) By previous Lemma, $\vdash_G \Gamma \Rightarrow \varphi$ if and only if $\vdash_{Gi+St} \Gamma \Rightarrow \varphi$. Thus, we only require to prove that $\vdash_{Gi+St} \Gamma \Rightarrow \varphi$ if and only if $\Gamma \vdash_N \varphi$.

On the one side, an intuitionistic sequent calculus derivation modulo stability axioms for

$\Gamma \Rightarrow \varphi$ will include some assumptions of the form $\Rightarrow \forall_x(\neg\neg\varphi_i \to \varphi_i)$, for formulas φ_i, with $i \leq k$ for some k in $\mathbb{N}$. Thus, by Theorem 13 there exists an intuitionistic proof in natural deduction using these stability axioms as assumptions. This intuitionistic natural derivation is converted into a classical derivation by including classical natural derivations for these assumptions.

On the other side, suppose that $\Gamma \vdash_N \varphi$ and let us assume that ∇ is a natural derivation for $\Gamma \vdash_N \varphi$ that uses only the classical rule $(\neg\neg_e)$; that is ∇ has no application of other exclusively classical rules such as (PBC) or (LEM). The derivation ∇ is transformed into an intuitionistic derivation with assumptions of stability axioms by applying to any application of the rule $(\neg\neg_e)$ in ∇, the following transformation:

$$\dfrac{\begin{matrix}\nabla' \\ \neg\neg\varphi\end{matrix}}{\varphi}\,(\neg\neg_e) \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\nabla' \\ \neg\neg\varphi\end{matrix} \qquad \dfrac{[\forall_x(\neg\neg\varphi \to \varphi)]^u}{\neg\neg\varphi \to \varphi}\,(\forall_e)}{\varphi}\,(\to_e)$$

In this manner, after replacing all applications of the rule $(\neg\neg_e)$, one obtains an intuitionistic natural derivation that has the original assumptions in Γ plus other assumptions that are stability axioms, say $\Gamma' = \forall_{x_1}(\neg\neg\varphi_1 \to \varphi_1), \ldots, \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k)$, for some k in $\mathbb{N}$. By Theorem 13 there exists an intuitionistic derivation a la Gentzen, say ∇'', for $\vdash_{Gi} \Gamma, \Gamma' \Rightarrow \varphi$. To conclude, note that one can get rid of all formulas in Γ' by using stability axioms of the form $\Rightarrow \forall_{x_i}(\neg\neg\varphi_i \to \varphi_i)$, for $i = 1, .., k$, and applications of the (Cut) rule as depicted below.

$$\dfrac{\Rightarrow \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k) \qquad \begin{matrix} \dfrac{\Rightarrow \forall_{x_1}(\neg\neg\varphi_1 \to \varphi_1) \qquad \begin{matrix}\nabla'' \\ \Gamma, \Gamma' \Rightarrow \varphi\end{matrix}}{\Gamma, \forall_{x_2}(\neg\neg\varphi_2 \to \varphi_2), \ldots, \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k) \Rightarrow \varphi}\,(\mathrm{Cut}) \\ \vdots \quad k \text{ applications of (Cut)} \\ \Gamma, \forall_{x_k}(\neg\neg\varphi_k \to \varphi_k) \Rightarrow \varphi \end{matrix}}{\Gamma \Rightarrow \varphi}\,(\mathrm{Cut})$$

This gives the desired derivation for $\vdash_{Gi+St} \Gamma \Rightarrow \varphi$. □

Exercise 50. *Prove all details of Theorem 14.*