Chapter 2

Derivations and Proofs in the Predicate Logic

2.1 Motivation

The propositional logic has several limitations for expressing ideas; mainly, it is not possible to quantify over sets of individuals and reason about them. These limitations can be better explained through examples:

“Every prime number bigger than 2 is odd”

“There exists a prime number greater than any given natural number”

In the language of the propositional logic this kind of properties can only be represented by a propositional variable because there is no way to split this information into simpler propositions joined by connectives and able to express the quantification over the natural numbers. In fact, the information in these sentences includes observations about sets of prime numbers, odd numbers, natural numbers, and quantification over them, and these relations cannot be straightforwardly captured in the language of propositional logic.

In order to overcome these limitations of the expressive power of the propositional logic, we extend its language with variables which range over individuals, and quantification over these variables. Thus, in this chapter we present the *predicate logic*, also known as *first-order logic*. In order to obtain a language with abilities to identify the required additional information, we need to extend the propositional language and provide a more expressive deductive calculus.

2.2 Syntax of the Predicate Logic

The language of the first-order predicate logic has two kinds of expressions: *terms* and *formulas.* While in the language of propositional logic formulas are built up from propositional variables, in the predicate logic they are built from atomic formulas, that are relational formulas expressing properties of terms such as "prime(2)", "prime(x)", "x is bigger than 2", etc. Formulas are built from relational formulas using the logical connectives as in the case of propositional logic, but in predicate logic also quantifiers over variables will be possible. Terms and basic relational formulas are built out of variables and two sets of symbols $\mathbb{F}$ and $\mathbb{P}$. Each function symbol in $\mathbb{F}$ and each predicate symbol in $\mathbb{P}$ comes with its fixed arity (that is, the number of its arguments). *Constants* can be seen as function symbols of arity zero. No predicate symbols with arity zero are allowed. This is the part of the language that is flexible since the sets $\mathbb{F}$ and $\mathbb{P}$ can be chosen arbitrarily.

Intuitively, predicates are functions that represent properties of terms. In order to define predicate formulas, we first define terms, and to do so, we assume an enumerable set $\mathbb{V}$ of *term variables.*

Definition 13 (Terms)**.** *A term t is defined inductively as follows:*

1. *Any variable $x \in \mathbb{V}$ is a term;*

2. *If $t_1, t_2, \ldots, t_n$ are terms, and $f \in \mathbb{F}$ is a function symbol with arity $n \geq 0$ then $f(t_1, t_2, \ldots, t_n)$ is a term. A function of arity zero is a constant.*

Notation 1. *We follow the usual notational convention for terms. Constant symbols, function symbols, arbitrary terms and variables are denoted by Roman lower-case letters, respectively, of the first, second, third and fourth quarters of the alphabet: $a, b, \ldots,$ for constant symbols; $f, g, \ldots,$ for function symbols; $s, t, \ldots$ for arbitrary terms and; $x, y, z,$ for variables.*

Terms, as given in the previous definition, could be equivalently presented by the following syntax:

$$t ::= x \;\; || \;\; f(t, \dots, t)$$

Definition 14 (Variable occurrence)**.** *The set of variables occurring in a term t, denoted by* $\mathtt{var}(t)$*, is inductively defined as follows:*

- *If $t = x$ then* $\mathtt{var}(t) = \{x\}$
- *If $t = f(t_1, \dots, t_n)$ then* $\mathtt{var}(t) = \mathtt{var}(t_1) \cup \dots \cup \mathtt{var}(t_n)$

We define the substitution of the term u for x in the term t, written $t[x/u]$, as the replacement of all occurrences of x in t by u. Formally, we have the following definition.

Definition 15 (Term Substitution)**.** *Let t, u be terms, and x, a variable. We define $t[x/u]$ inductively as follows:*

- *$x[x/u] = u$;*
- *$y[x/u] = y$, for $y \neq x$;*
- *$f(t_1, \dots, t_n)[x/u] = f(t_1[x/u], \dots, t_n[x/u])$ $(n \geq 0)$.*

Now we are a ready to define the formulas of the predicate logic:

Definition 16 (Formulas)**.** *The set of formulas of the first-order predicate logic over a variable set $\mathbb{V}$ and a symbol set $S = (\mathbb{F}, \mathbb{P})$ is inductively defined as follows:*

1. *$\bot$ and $\top$ are formulas;*
2. *If $p \in \mathbb{P}$ with arity $n > 0$, and $t_1, t_2, \dots, t_n$ are terms then $p(t_1, t_2, \dots, t_n)$ is a formula;*
3. *If φ is a formula then so is $(\neg\varphi)$;*
4. *If φ_1 and φ_2 are formulas then so are $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$ and $(\varphi_1 \rightarrow \varphi_2)$;*

5. *If $x \in \mathbb{V}$ and φ is a formula then $(\forall_x \varphi)$ and $(\exists_x \varphi)$ are formulas.*

The symbol $\forall_x$ (resp. $\exists_x$) means "for all x" (resp. "there exists a x"), and the formula φ is the *body* of the formula $(\forall_x \varphi)$ (resp. $(\exists_x \varphi)$). Since quantification is restricted to variable terms, the defined language corresponds to a so called *first-order language*.

The set of formulas of the predicate logic have the following syntax:

$$\varphi ::= p(t, \ldots, t) \;||\; \bot \;||\; \top \;||\; (\neg \varphi) \;||\; (\varphi \wedge \varphi) \;||\; (\varphi \vee \varphi) \;||\; (\varphi \rightarrow \varphi) \;||\; (\forall_x \varphi) \;||\; (\exists_x \varphi)$$

Formulas of the form $p(t_1, \ldots, t_n)$ are called *atomic formulas* because they cannot be decomposed into simpler formulas. As usual, parenthesis are used to avoid ambiguities and the external ones will be omitted. The quantifiers $\forall_x$ and $\exists_x$ bind the variable x in the body of the formula. This idea is formalized by the notion of *scope of a quantifier*:

Definition 17 (Scope of quantifiers, free and bound variables)**.** *The scope of $\forall_x$ (resp. $\exists_x$) in the formula $\forall_x \varphi$ (resp. $\exists_x \varphi$) is the body of the quantified formula: φ. An occurrence of a variable x in the scope of $\forall_x$ or $\exists_x$ is called bound. An occurrence of a variable that is not bound is called free.*

Since the body of a quantified formula can have occurrences of other quantified formulas that abstract the same variable symbol, it is necessary to provide more precise mechanisms to build the sets of free and bound variables of a predicate formula. This can be done inductively according to the following definitions.

Definition 18 (Construction of the set of free variable)**.** *Let φ be a formula of the predicate logic. The set of free variables of φ, denoted by $\mathtt{fv}(\varphi)$, is inductively defined as follows:*

1. $\mathtt{fv}(\bot) = \mathtt{fv}(\top) = \emptyset$;

2. $\mathtt{fv}(p(t_1, \ldots, t_n)) = \mathtt{var}(t_1) \cup \ldots \cup \mathtt{var}(t_n)$;

3. $\mathtt{fv}(\neg \varphi) = \mathtt{fv}(\varphi)$;

4. $\mathtt{fv}(\varphi \Box \psi) = \mathtt{fv}(\varphi) \cup \mathtt{fv}(\psi)$, *where* $\Box \in \{\wedge, \vee, \rightarrow\}$;

5. $\mathtt{fv}(Q_x\varphi) = \mathtt{fv}(\varphi) \setminus \{x\}$, *where* $Q \in \{\forall, \exists\}$.

A formula without occurrences of free variables is called a sentence.

Definition 19 (Construction of the set of bound variables)**.** *Let* φ *be a formula of the predicate logic. The set of bound variables of* φ*, denoted by* $\mathtt{bv}(\varphi)$*, is inductively defined as follows:*

1. $\mathtt{bv}(\bot) = \mathtt{bv}(\top) = \emptyset$;

2. $\mathtt{bv}(p(t_1, \ldots, t_n)) = \emptyset$;

3. $\mathtt{bv}(\neg\varphi) = \mathtt{bv}(\varphi)$;

4. $\mathtt{bv}(\varphi\Box\psi) = \mathtt{bv}(\varphi) \cup \mathtt{bv}(\psi)$, *where* $\Box \in \{\wedge, \vee, \rightarrow\}$;

5. $\mathtt{bv}(Q_x\varphi) = \mathtt{bv}(\varphi) \cup \{x\}$, *where* $Q \in \{\forall, \exists\}$.

Informally, the name of a bound variable is not important in the sense that it can be *renamed* to any *fresh* name without changing the semantics of the term. For instance, the formulas $\forall_x(x \leq x)$, $\forall_y(y \leq y)$ and $\forall_z(z \leq z)$ represent the very same object. The sole restriction that needs to be considered is that *variable capture* is forbidden, i.e. no free variable can become bound after a renaming of a variable. For instance, if p denotes a binary predicate then $\forall_x p(x, y)$ is a renaming of $\forall_z p(z, y)$, while $\forall_y p(y, y)$ is not. The next definition will formalize the notion of substitution. The capture of free variables by a substitution is also forbidden, and we assume that a renaming of bound variables is always performed when necessary to avoid capture.

Definition 20 (Substitution)**.** *Let* φ *be a formula of the predicate logic. The substitution of* x *by* t *in* φ*, written* $\varphi[x/t]$*, is inductively defined as follows:*

1. $\bot[x/t] = \bot$ *and* $\top[x/t] = \top$;

2. $p(t_1, \ldots, t_n)[x/t] = p(t_1[x/t], \ldots, t_n[x/t])$;

3. $(\neg\psi)[x/t] = \neg(\psi[x/t])$;

4. $(\psi \Box \gamma)[x/t] = (\psi[x/t])\Box(\gamma[x/t])$, *where* $\Box \in \{\wedge, \vee, \rightarrow\}$*;*

5. $(Q_y\psi)[x/t] = Q_y(\psi[x/t])$, *where* $Q \in \{\exists, \forall\}$*, and renaming of bound variables is assumed to avoid capture of variables.*

Example 6. *Consider the following applications of substitution:*

- $(\forall_x p(y))[y/x] = \forall_z p(y)[y/x] = \forall_z p(y[y/x]) = \forall_z p(x)$ *and*
- $(\forall_x p(x))[x/t] = \forall_y p(y)[x/t] = \forall_y p(y[x/t]) = \forall_y p(y)$.

Notice that in the second application, renaming x as y was necessary to avoid capture.

The necessary renamings to avoid capture of variables in substitutions can be implemented in several ways. For instance, it can be done by modifying item *5* in the definition of substitution in such a way that before propagating the substitution inside the scope of a quantified formula of the form $(Q_x\varphi)[x/t]$, where $Q \in \{\forall, \exists\}$, it is checked whether $x = y$ or $x \in \mathtt{fv}(t)$: whenever $x = y$ or $x \in \mathtt{fv}(t)$ renaming the quantified variable name x as a fresh variable name z is applied, in other case no renaming is needed:

$$(Q_x\varphi)[y/t] = \begin{cases} (Q_z\varphi[x/z][y/t]), & \text{if } x = y \text{ or } x \in \mathtt{fv}(t), \\ (Q_x\varphi[y/t]), & \text{otherwise.} \end{cases}$$

The size of predicate expressions (terms and formulas) is defined in the usual manner.

Definition 21 (Size of predicate expressions)**.** *Let t be a predicate term and φ a predicate formula. The size of t, denoted as $|t|$, is recursively defined as below.*

- $|x| = 1$, *for* $x \in \mathbb{V}$*;*
- $|f(t_1, \ldots, t_n)| = 1 + |t_1| + \cdots + |t_n|$, *for* $n \geq 0$.

The size of φ, denoted as $|\varphi|$, is recursively defined as below.

- $|\bot| = |\top| = 1$*;*
- $|p(t_1, \ldots, t_n)| = 1 + |t_1| + \cdots + |t_n|$, *for* $n \geq 1$*;*

- $|(\neg\psi)| = 1 + |\psi|$;
- $|(\psi\Box\gamma)| = 1 + |\psi| + |\gamma|$, *where* $\Box \in \{\wedge, \vee, \rightarrow\}$;
- $|(Q_y\psi)| = 1 + |\psi|$, *where* $Q \in \{\exists, \forall\}$.

Exercise 23.

a. *Consider a predicate formula* φ *and a term* t. *Prove that there are no bound variables in the new occurrences of* t *in the formula* $\varphi[x/t]$. *For doing this use induction on the structure of* φ. *Of course, occurrences of the term* t *in the original formula* φ *might be under the scope of quantifiers and consequently variables occurring in these subterms would be bound.*

b. *Let* k *be the number of free occurrences of the variable* x *in the predicate formula* φ. *Prove, also by induction on* φ, *that the size of the term* $\varphi[x/t]$ *is given by* $k|t| + |\varphi| - k$.

c. *For* $x \neq y$, *prove also that:*

   i. $\varphi[x/s][x/t] = \varphi[x/s[x/t]]$;

   ii. $\varphi[x/s][y/t] = \varphi[x/s[y/t]][y/t]$, *if* $y \notin \mathtt{var}(t)$;

   iii. $\varphi[x/s][y/t] = \varphi[y/t][x/s]$, *if* $x \notin \mathtt{var}(t)$ *and* $y \notin \mathtt{var}(s)$.

2.3 Natural Deduction in the Predicate Logic

The set of rules of natural deduction for the predicate logic is an extension of the set presented for the propositional logic. The rules for conjunction, disjunction, implication and negation have the same shape, but note that now the formulas are the ones of predicate logic. In this section, we also discuss the minimal, intuitionistic and classical predicate logic. Thus the rules are those in Tables 1.2, without the rule $(\bot_e)$ for the minimal predicate logic and with this rule for the intuitionistic predicate logic, and in Table 1.3 for the classical predicate logic, plus four additional rules for dealing with quantified formulas.

We start by expanding the set of natural deduction rules with the ones for quantification. The first one is the elimination rule for the universal quantifier:

$$\frac{\forall_x \varphi}{\varphi[x/t]}\ (\forall_e)$$

The intuition behind this rule is that from a proof of $\forall_x \varphi$, we can conclude $\varphi[x/t]$, where t is any term. This transformation is done by the substitution operator previously defined, that replaces every free occurrence of x by an arbitrary term t in φ. According to the substitution operator, "every" occurrence of x in φ is replaced with the "same" term t. The following example shows an application of $(\forall_e)$ in a derivation.

Example 7. $\forall_x p(a,x), \forall_x \forall_y (p(x,y) \to p(f(x),y)) \vdash p(f(a),f(a))$.

$$\dfrac{\dfrac{\forall_x p(a,x)}{p(a,f(a))}\ (\forall_e) \qquad \dfrac{\dfrac{\forall_x \forall_y (p(x,y) \to p(f(x),y))}{\forall_y p(a,y) \to p(f(a),y)}\ (\forall_e)}{p(a,f(a)) \to p(f(a),f(a))}\ (\forall_e)}{p(f(a),f(a))}\ (\to_e)$$

Note that the application of $(\to_e)$ is identical to what is done in the propositional calculus, except from the fact that now it is applied to predicate formulas.

The introduction rule for the universal quantifier is more subtle. In order to prove $\forall_x \varphi$ one needs first to prove $\varphi[x/x_0]$ in such a way that no open assumption in the derivation of $\varphi[x/x_0]$ can contain occurrences of x_0. This restriction is necessary to guarantee that x_0 is general enough and can be understood as "any" term, i.e. nothing has been assumed concerning x_0. The $(\forall_i)$ rule is given by:

$$\frac{\varphi[x/x_0]}{\forall_x \varphi}\ (\forall_i)$$

where x_0 is a fresh variable not occurring in any open assumption in the derivation of $\varphi[x/x_0]$.

Example 8. $\forall_x(p(x) \wedge q(x)) \vdash \forall_x (p(x) \rightarrow q(x))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall_x(p(x) \wedge q(x))}{p(x_0) \wedge q(x_0)}\ (\forall_e)}{q(x_0)}\ (\wedge_e)}{p(x_0) \rightarrow q(x_0)}\ (\rightarrow_i)\emptyset}{\forall_x(p(x) \rightarrow q(x))}\ (\forall_i)$$

Note that the formula $p(x_0) \rightarrow q(x_0)$ depends only on the hypothesis $\forall_x(p(x) \wedge q(x))$, which does not contain x_0. Thus x_0 might be considered arbitrary, which allows the generalization through application of rule $(\forall_i)$. In fact, note that the above proof of $p(x_0) \rightarrow q(x_0)$ could be done for any other term, say t instead x_0, which explains the generality of x_0 in the above example.

The introduction rule for the existential quantifier is as follows:

$$\frac{\varphi[x/t]}{\exists_x\, \varphi}\ (\exists_i)$$

where t is any term.

Example 9. $\forall_x\, q(x) \vdash \exists_x\, q(x)$.

$$\dfrac{\dfrac{\forall_x\, q(x)}{q(x_0)}\,(\forall_e)}{\exists_x\, q(x)}\,(\exists_i)$$

Similarly to $(\forall_i)$, the elimination rule for the existential quantifier is more subtle:

$$\dfrac{\exists_x\, \varphi \qquad \begin{array}{c}[\varphi[x/x_0]]^u\\ \vdots \\ \chi\end{array}}{\chi}\,(\exists_e)\ u$$

This rule requires the variable x_0 be a fresh variable neither occurring in any other open assumption than in $[\varphi[x/x_0]]^u$ itself nor in the conclusion $\texttt{fv}(\chi)$. The intuition of this rule might be explained as follows: knowing that $\exists_x\, \varphi$ holds, if assuming that an arbitrary x_0 witnesses the property φ, i.e., assuming $[\varphi[x/x_0]]^u$, one can infer χ, then χ holds in general. This kind of analysis is done, for instance, when properties about numbers are inferred from the knowledge of the existence of prime numbers of arbitrary size, or (good/bad) properties about institutions are inferred from the knowledge of the existence of the (good/bad) qualities of some individuals in their staffs. These general properties are inferred without knowing specific prime numbers or without knowing who are specifically the (good/bad) individuals in the institutions.

Example 10. *This example attempts to bring a little bit intuition about the use of these rules. Let p, q and r be predicate symbols with the intended meanings: $p(z)$ means "z is a planet different from the earth with similar characteristics"; $q(y)$ means "country y adopts action to mitigate global warming" and $r(x, y)$ means "x is a leader, who works in the ministry of agriculture or environment of country y and who is worried about climate change". Thus,* *from the hypotheses $\forall_y\exists_x r(x,y)$, $\forall_y\forall_x(r(x,y)\to q(y))$ and $\forall_z(\forall_y q(y)\to\neg p(z))$, we can infer that we do not need a "Planet B" as below.*

$$\dfrac{\dfrac{\dfrac{\forall_y\exists_x r(x,y)}{\exists_x r(x,c_0)}(\forall_e) \qquad \dfrac{[r(l_0,c_0)]^u \qquad \dfrac{\dfrac{\forall_y\forall_x(r(x,y)\to q(y))}{\forall_x(r(x,c_0)\to q(c_0))}(\forall_e)}{r(l_0,c_0)\to q(c_0)}(\forall_e)}{q(c_0)}(\to_e)}{\dfrac{q(c_0)}{\forall_y q(y)}(\forall_e)}(\exists_e)\,u \qquad \dfrac{\forall_z(\forall_y q(y)\to\neg p(z))}{\forall_y q(y)\to\neg p(B)}(\forall_e)}{\neg p(B)}(\to_e)$$

Example 11. *The use of substitution in natural deduction rules for quantifiers is illustrated in this example. Initially, consider a unary predicate p. Below, it is depicted a derivation for $\exists_x\, p(x) \vdash \neg\forall_x\,\neg p(x)$.*

$$\dfrac{\exists_x p(x) \qquad \dfrac{\dfrac{[p(x_0)]^u \qquad \dfrac{[\forall_x\neg p(x)]^v}{\neg p(x_0)}(\forall_e)}{\bot}(\neg_e)}{\neg\forall_x\neg p(x)}(\neg_i)\,v}{\neg\forall_x\neg p(x)}(\exists_e)\,u$$

Now, consider a predicate formula φ and a variable x that might or might not occur free in φ. The next derivation, denoted as ∇_3, proofs that $\vdash \exists_x\varphi\to\neg\forall_x\neg\varphi$. Despite the proof for φ appears to be the same than the one above for the unary predicate p, several subtle points should be highlighted. In the application of rule $(\exists_e)$ in the derivation ∇_3, it is forbidden the selection of a witness variable "y", to be used in the witness assumption $[\varphi[x/y]]^w$, such that y belongs to the set of free variables occurring in φ. Indeed, y should be a fresh variable. To *understand this restriction, consider $\varphi = q(y,x)$ and suppose the intended meaning of q is "x is the double of y". If the existential formula is $\exists_x p(y,x)$ the witness assumption cannot be $p(y,x)[x/y] = p(y,y)$, since this selection of "y" is not arbitrary.*

$$\dfrac{\dfrac{\dfrac{[\exists_x\varphi]^u \qquad \dfrac{[\varphi[x/y]]^w \qquad \dfrac{[\forall_x\neg\varphi]^v}{\neg\varphi[x/y]}\,(\forall_e)}{\bot}\,(\neg_e)}{\bot}\,(\exists_e)\ w}{\neg\forall_x\neg\varphi}\,(\neg_i)\ v}{\exists_x\varphi \to \neg\forall_x\neg\varphi}\,(\to_i)\ u$$

The rules for quantification discussed so far, are summarized in Table 2.1. These rules together with the deduction rules for introduction and elimination of the connectives: $\wedge, \vee, \neg$ and $\to$, conform the set of natural deduction rules for the minimal predicate logic (that is, rules in Tables 2.1 and 1.2 except rule $(\bot_e)$) . If in addition, we include the intuitionistic absurdity rule, we obtain the natural deduction calculus for the intuitionistic predicate logic (that is all rules in Tables 2.1 and 1.2). The classical predicate calculus is obtained from the intuitionistic one, changing the intuitionistic absurdity rule by the rule (PBC) (that is, rules in Tables 2.1 and 1.3).

Table 2.1: Natural deduction rules for quantification

introduction rules	elimination rules
$\dfrac{\varphi[x/x_0]}{\forall_x\varphi}\ (\forall_i)$ where x_0 cannot occur free in any open assumption.	$\dfrac{\forall_x\varphi}{\varphi[x/t]}\ (\forall_e)$
$\dfrac{\varphi[x/t]}{\exists_x\varphi}\ (\exists_i)$	$\dfrac{\exists_x\varphi \qquad \begin{matrix}[\varphi[x/x_0]]^u\\ \vdots \\ \chi\end{matrix}}{\chi}\ (\exists_e)\ u$ where x_0 cannot occur free in any open assumption on the right and in χ.

Example 12. *The sequent* $\vdash \exists_x\neg\varphi \to \neg\forall_x\varphi$ *has the following intuitionistic proof* ∇_1*:*

$$\dfrac{\dfrac{\dfrac{[\exists_x\neg\varphi]^u \qquad \dfrac{\dfrac{[\forall_x\varphi]^v}{\varphi[x/y]}\ (\forall_e) \qquad [\neg\varphi[x/y]]^w}{\bot}\ (\neg_e)}{\bot}\ (\exists_e)\ w}{\neg\forall_x\varphi}\ (\neg_i)\ v}{\exists_x\neg\varphi \to \neg\forall_x\,\varphi}\ (\to_i)\ u$$

The proof ∇_1 *can be used to prove the sequent* $\vdash \forall_x\varphi \to \neg\exists_x\neg\varphi$ *as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\nabla_1}{\exists_x\neg\varphi \to \neg\forall_x\varphi} \qquad [\exists_x\neg\varphi]^v}{\neg\forall_x\varphi}\ (\to_e) \qquad [\forall_x\varphi]^w}{\bot}\ (\neg_e)}{\neg\exists_x\neg\varphi}\ (\neg_i)\ v}{\forall_x\varphi \to \neg\exists_x\neg\varphi}\ (\to_i)\ w$$

Exercise 24. *Prove intuitionistically that* $\neg\exists_x\varphi \dashv\vdash \forall_x\neg\varphi$.

Exercise 25. *Prove that:*

a. if x *does not occur free in* ψ *then prove that* $(\exists_x\phi) \to \psi \vdash \forall_x(\phi \to \psi)$*; and*

b. if x *does not occur free in* ψ *then prove that* $(\forall_x\phi) \to \psi \vdash \exists_x(\phi \to \psi)$.

Exercise 26. *Prove that:*

a. $(\forall_x\phi) \wedge (\forall_x\psi) \dashv\vdash \forall_x(\phi \wedge \psi)$*; and*

b. $(\exists_x\phi) \vee (\exists_x\psi) \dashv\vdash \exists_x(\phi \vee \psi)$.

Exercise 27. *Prove that* $\forall_x(p(x) \to \neg q(x)) \vdash \neg(\exists_x(p(x) \wedge q(x)))$.

The interpretation of formulas in the classical logic is different from the one in the intuitionistic logic. While in the intuitionistic logic the goal is to "have a constructive proof" of a formula φ, in the classical logic the goal is to "establish a proof of the truth" of φ. For instance, a classical proof admits the truth of a formula of the form $\exists_x\varphi$ without having an explicit witness for x. Such kind of proof (without an explicit witness for the existential) is not accepted in the intuitionistic logic. As an example, suppose that one wants to prove that there exists two irrational numbers x and y such that x^y is rational. If $r(x)$ means that "x is a rational number" then one aims to prove the sequent $\vdash \exists_x\exists_y(\neg r(x) \wedge \neg r(y) \wedge r(x^y))$. In order to do so, we assume some obvious facts in algebra, such as $\neg r(\sqrt{2})$ and $r((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})$.

$$\dfrac{(\text{LEM})\,\dfrac{}{r(\sqrt{2}^{\sqrt{2}}) \vee \neg r(\sqrt{2}^{\sqrt{2}})} \qquad \nabla_1 \qquad \nabla_2}{\exists_x\exists_y(\neg r(x) \wedge \neg r(y) \wedge r(x^y))}\,(\vee_e)\ a, b$$

where ∇_1 is given by

$$\dfrac{\dfrac{\dfrac{\neg r(\sqrt{2}) \qquad \dfrac{\neg r(\sqrt{2}) \qquad [r((\sqrt{2})^{\sqrt{2}})]^a}{\neg r(\sqrt{2}) \wedge r((\sqrt{2})^{\sqrt{2}})}\,(\wedge_i)}{\neg r(\sqrt{2}) \wedge \neg r(\sqrt{2}) \wedge r((\sqrt{2})^{\sqrt{2}})}\,(\wedge_i)}{\exists_y(\neg r(\sqrt{2}) \wedge \neg r(y) \wedge r((\sqrt{2})^y))}\,(\exists_i)}{\exists_x\exists_y(\neg r(x) \wedge \neg r(y) \wedge r(x^y))}\,(\exists_i)$$

and ∇_2 is given by

$$\dfrac{\dfrac{\dfrac{[\neg r(\sqrt{2}^{\sqrt{2}})]^b \qquad \dfrac{\neg r(\sqrt{2}) \qquad r((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})}{\neg r(\sqrt{2}) \wedge r((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})}\,(\wedge_i)}{\neg r(\sqrt{2}^{\sqrt{2}}) \wedge \neg r(\sqrt{2}) \wedge r((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})}\,(\wedge_i)}{\exists_y(\neg r(\sqrt{2}^{\sqrt{2}}) \wedge \neg r(y) \wedge r((\sqrt{2}^{\sqrt{2}})^y))}\,(\exists_i)}{\exists_x\exists_y(\neg r(x) \wedge \neg r(y) \wedge r(x^y))}\,(\exists_i)$$

In the proof above, the witnesses depend on whether $\sqrt{2}^{\sqrt{2}}$ is rational or not. In the positive case, taking $x = y = \sqrt{2}$ allows us to conclude that x^y is rational, and in the negative case, this conclusion is achieved by taking $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. So we proved the "existence" of an object without knowing explicitly the witnesses for x and y. This is acceptable as a proof in the classical logic, but not in the intuitionistic one.

Analogously to the intuitionistic case, the rules of the classical predicate logic are given by the rule schemes for the connectives ($\wedge, \vee, \neg$ and $\rightarrow$), the classical absurdity rule (PBC) (see Table 1.3) and the rules for the quantifiers (Table 2.1).

Example 13. *While the sequents $\vdash \exists_x\varphi \rightarrow \neg\forall_x\neg\varphi$ and $\vdash \forall_x\varphi \rightarrow \neg\exists_x\neg\varphi$ have intuitionistic (indeed minimal) proofs as shown in Examples 11 and 12, the sequents $\vdash \neg\exists_x\neg\varphi \rightarrow \forall_x\varphi$ and $\vdash \neg\forall_x\neg\varphi \rightarrow \exists_x\varphi$ have only classical proofs. A proof for the former is given below.*

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\exists x\neg\varphi]^u \qquad \dfrac{[\neg\varphi[x/y]]^v}{\exists_x\neg\varphi}\,(\exists_i)}{\perp}\,(\neg_e)}{\varphi[x/y]}\,(\mathrm{PBC})\ v}{\forall_x\varphi}\,(\forall_i)}{\neg\exists_x\neg\varphi\to\forall_x\varphi}\,(\to_i)\ u$$

Moreover, note that the above proof jointly with the one given in Example 11 shows that $\forall_x\varphi \dashv\vdash \neg\exists_x\neg\varphi$.

A proof of the sequent $\vdash \neg\forall_x\neg\varphi\to\exists_x\varphi$ *is given below.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi[x/y]]^v}{\exists_x\varphi}\,(\exists_i) \qquad [\neg\exists_x\varphi]^u}{\perp}\,(\neg_e)}{\neg\varphi[x/y]}\,(\neg_i)\ v}{\forall_x\neg\varphi}\,(\forall_i) \qquad [\neg\forall_x\neg\varphi]^w}{\perp}\,(\neg_e)}{\exists_x\varphi}\,(\mathrm{PBC})\ u}{\neg\forall_x\neg\varphi\to\exists_x\varphi}\,(\to_i)\ w$$

Finally, this proof jointly with the one given in Example 12 shows that $\exists_x\varphi \dashv\vdash \neg\forall_x\neg\varphi$.

To verify that there are no possible intuitionistic derivations, notice that $\neg\exists_x\neg\varphi\to\forall_x\varphi$ *and* $\neg\forall_x\neg\varphi\to\exists_x\varphi$ *together with the intuitionistic (indeed minimal) deduction rules allows derivation of non intuitionistic theorems such as* $\neg\neg\varphi\vdash\varphi$ *(see next Exercise 28).*

Exercise 28. *Prove that there exist derivations for* $\neg\neg\varphi\vdash\varphi$ *using only the minimal natural* *deduction rules and each of the assumptions:*

a. $\neg\exists_x\neg\varphi \to \forall_x\varphi$ *and*

b. $\neg\forall_x\neg\varphi \to \exists_x\varphi$.

Hint: you can choose the variable x *as any variable that does not occurs in* φ. *Thus, the application of rule* $(\exists_e)$ *over the existential formula* $\exists_x\varphi$ *has as witness assumption* $[\varphi[x/x_0]]^w$ *that has no occurrences of* x_0.

In Exercise 24 we prove that there are intuitionistic derivations for $\neg\exists_x\varphi \dashv\vdash \forall_x\neg\varphi$. Also, in Example 12 we give an intuitionistic derivation for $\exists_x\neg\varphi \vdash \neg\forall_x\varphi$. Indeed, one can obtain minimal derivations for these three sequents.

Exercise 29. *To complete* $\neg\forall_x\varphi \dashv\vdash \exists_x\neg\varphi$ *(see Example 12), prove that* $\neg\forall_x\varphi \vdash \exists_x\neg\varphi$.

2.4 Semantics of the Predicate Logic

As done for the propositional logic in Chapter 1, here we present the standard semantics of first-order classical logic. The semantics of the predicate logic is not a direct extension of the one of propositional logic. Although this is not surprising, since the predicate logic has a richer language, there are some interesting points concerning the differences between propositional and predicate semantics that will be examined in this section. In fact, while a propositional formula has only finitely many interpretations, a predicate formula can have infinitely many ones.

We start with an example: let p be a unary predicate symbol, and consider the formula $\forall_x p(x)$. The variable x ranges over a domain, say the set of natural numbers $\mathbb{N}$. Is this formula true or false? Certainly, it depends on how the predicate symbol p is interpreted. If one interprets $p(x)$ as "x is a prime number", then it is false, but if $p(x)$ means that "x is a natural number" then it is true. Observe that the interpretation depends on the chosen domain, and hence the latter interpretation of p will be false over the domain of integers $\mathbb{Z}$.

This situation is similar in the propositional logic: according to the interpretation, some formulas can be either true or false. So what do we need to determine the truth value of a