Universidade de Brasília - Instituto de Ciências Exatas
Departamento de Ciência da Computação
Lógica Computacional 1 **(2022/1)**

28 de julho de 2022

Lista: Dedução Natural (Gabarito)

Em adição aos exercícios que aparecem nas notas de aula, solucione os listados a seguir. Nas sua derivações, sempre indique qual regra dedutiva é utilizada em cada passo.

1. Prove os sequentes a seguir utilizando apenas a lógica proposicional minimal:

   (a) $\neg\neg(\phi \wedge \psi) \dashv\vdash \neg\neg\phi \wedge \neg\neg\psi$.
   Solução

$$\dfrac{\dfrac{\dfrac{\neg\neg(\phi\wedge\psi) \qquad \dfrac{\dfrac{[\neg\phi]^u \qquad \dfrac{[\phi\wedge\psi]^v}{\phi}(\wedge_e)}{\bot}(\neg_e)}{\neg(\phi\wedge\psi)}(\neg_i)\ v}{\bot}(\neg_e)}{\neg\neg\phi}(\neg_i)\ u \qquad \dfrac{\dfrac{\neg\neg(\phi\wedge\psi) \qquad \dfrac{\dfrac{[\neg\psi]^x \qquad \dfrac{[\phi\wedge\psi]^y}{\psi}(\wedge_e)}{\bot}(\neg_e)}{\neg(\phi\wedge\psi)}(\neg_i)\ y}{\bot}(\neg_e)}{\neg\neg\psi}(\neg_i)\ x}{\neg\neg\phi\wedge\neg\neg\psi}(\wedge_i)$$

$$\dfrac{\dfrac{(\wedge_e)\dfrac{(\neg\neg\phi)\wedge(\neg\neg\psi)}{\neg\neg\psi} \qquad \dfrac{\dfrac{(\wedge_e)\dfrac{(\neg\neg\phi)\wedge(\neg\neg\psi)}{\neg\neg\phi} \qquad \dfrac{\dfrac{(\wedge_i)\dfrac{[\phi]^x \qquad [\psi]^y}{\phi\wedge\psi} \qquad [\neg(\phi\wedge\psi)]^z}{\bot}(\neg_e)}{\neg\phi}(\neg_i)\ x}{\bot}(\neg_e)}{\neg\psi}(\neg_i)\ y}{\bot}(\neg_e)}{\neg\neg(\phi\wedge\psi)}(\neg_i)\ z$$

   (b) $\neg(\phi \vee \psi) \dashv\vdash \neg\phi \wedge \neg\psi$.
   Solução

$$\dfrac{(\neg_i)\ u\ \dfrac{(\neg_e)\ \dfrac{\neg(\phi\vee\psi) \qquad (\vee_i)\dfrac{[\phi]^u}{\phi\vee\psi}}{\bot}}{\neg\phi} \qquad \dfrac{\dfrac{\neg(\phi\vee\psi) \qquad \dfrac{[\psi]^v}{\phi\vee\psi}(\vee_i)}{\bot}(\neg_e)}{\neg\psi}(\neg_i)\ v}{\neg\phi\wedge\neg\psi}(\wedge_i)$$

$$\dfrac{\dfrac{[(\phi\vee\psi)]^u \qquad \dfrac{(\wedge_e)\,\dfrac{(\neg\phi\wedge\neg\psi)}{\neg\phi} \qquad [\phi]^x}{\bot}(\neg_e) \qquad \dfrac{\dfrac{(\neg\phi\wedge\neg\psi)}{\neg\psi}(\wedge_e) \qquad [\psi]^y}{\bot}(\neg_e)}{\bot}(\vee_e)\ x,y}{\neg(\phi\vee\psi)}(\neg_i)\ u$$

(c) $(\phi\wedge\psi)\wedge\varphi \dashv\vdash \phi\wedge(\psi\wedge\varphi)$.

Solução

$$\dfrac{(\wedge_e)\,\dfrac{(\wedge_e)\,\dfrac{(\phi\wedge\psi)\wedge\varphi}{(\phi\wedge\psi)}}{\phi} \qquad \dfrac{(\wedge_e)\,\dfrac{(\wedge_e)\,\dfrac{(\phi\wedge\psi)\wedge\varphi}{\phi\wedge\psi}}{\psi} \qquad \dfrac{(\phi\wedge\psi)\wedge\varphi}{\varphi}(\wedge_e)}{(\psi\wedge\varphi)}(\wedge_i)}{\phi\wedge(\psi\wedge\varphi)}(\wedge_i)$$

$$\dfrac{\dfrac{(\wedge_e)\,\dfrac{\phi\wedge(\psi\wedge\varphi)}{\phi} \qquad \dfrac{\dfrac{\phi\wedge(\psi\wedge\varphi)}{(\psi\wedge\varphi)}(\wedge_e)}{\psi}(\wedge_e)}{(\phi\wedge\psi)}(\wedge_i) \qquad \dfrac{\dfrac{\phi\wedge(\psi\wedge\varphi)}{(\psi\wedge\varphi)}(\wedge_e)}{\varphi}(\wedge_e)}{(\phi\wedge\psi)\wedge\varphi}(\wedge_i)$$

(d) $(\phi\vee\psi)\vee\varphi \dashv\vdash \phi\vee(\psi\vee\varphi)$.

Solução

$$\dfrac{(\phi\vee\psi)\vee\varphi \qquad \dfrac{[(\phi\vee\psi)]^x \qquad \dfrac{[\phi]^z}{\phi\vee(\psi\vee\varphi)}(\vee_i) \qquad \dfrac{\dfrac{[\psi]^w}{(\psi\vee\varphi)}(\vee_i)}{\phi\vee(\psi\vee\varphi)}(\vee_i)}{\phi\vee(\psi\vee\varphi)}(\vee_e)\ z,w \qquad \dfrac{\dfrac{[\varphi]^y}{(\psi\vee\varphi)}(\vee_i)}{\phi\vee(\psi\vee\varphi)}(\vee_i)}{\phi\vee(\psi\vee\varphi)}(\vee_e)\ x,y$$

$$\dfrac{\phi\vee(\psi\vee\varphi) \qquad \dfrac{\dfrac{[\phi]^x}{(\phi\vee\psi)}(\vee_i)}{(\phi\vee\psi)\vee\varphi}(\vee_i) \qquad \dfrac{[(\psi\vee\varphi)]^y \qquad \dfrac{\dfrac{[\psi]^u}{(\phi\vee\psi)}(\vee_i)}{(\phi\vee\psi)\vee\varphi}(\vee_i) \qquad \dfrac{[\varphi]^v}{(\phi\vee\psi)\vee\varphi}(\vee_i)}{(\phi\vee\psi)\vee\varphi}(\vee_e)\ u,v}{(\phi\vee\psi)\vee\varphi}(\vee_e)\ x,y$$

(e) $\phi \to \psi \vdash \delta \lor \phi \to \delta \lor \psi$

Solução

$$\dfrac{\dfrac{[\delta\lor\phi]^z \qquad \dfrac{\dfrac{\phi\to\psi \qquad [\phi]^x}{\psi}\,(\to_e)}{\psi\lor\delta}\,(\lor_i) \qquad \dfrac{[\delta]^y}{\psi\lor\delta}\,(\lor_i)}{\psi\lor\delta}\,(\lor_e)\ x,y}{\delta\lor\phi\to\delta\lor\psi}\,(\to_i)\ z$$

(f) $(\delta \land \phi) \lor (\delta \land \psi) \dashv\vdash \delta \land (\phi \lor \psi)$

Solução

$$\dfrac{\dfrac{(\delta\land\phi)\lor(\delta\land\psi) \quad \dfrac{[\delta\land\phi]^u}{\delta}\,(\land_e) \quad \dfrac{[\delta\land\psi]^v}{\delta}\,(\land_e)}{\delta}\,(\lor_e)\ u,v \qquad \dfrac{(\delta\land\phi)\lor(\delta\land\psi) \quad \dfrac{\dfrac{[\delta\land\phi]^x}{\phi}\,(\land_e)}{\phi\lor\psi}\,(\lor_i) \quad \dfrac{\dfrac{[\delta\land\psi]^t}{\psi}\,(\land_e)}{\phi\lor\psi}\,(\lor_i)}{\phi\lor\psi}\,(\lor_e)\ x,t}{\delta\land(\phi\lor\psi)}\,\land i$$

$$\dfrac{\dfrac{\delta\land(\phi\lor\psi)}{\phi\lor\psi}\,(\land_e) \qquad \dfrac{\dfrac{\dfrac{\delta\land(\phi\lor\psi)}{\delta}\,(\land_e) \qquad [\phi]^x}{\delta\land\phi}\,(\land_i)}{(\delta\land\phi)\lor(\delta\land\psi)}\,(\lor_i) \qquad \dfrac{\dfrac{\dfrac{\delta\land(\phi\lor\psi)}{\delta}\,(\land_e) \qquad [\psi]^y}{\delta\land\psi}\,(\land_i)}{(\delta\land\phi)\lor(\delta\land\psi)}\,(\lor_i)}{(\delta\land\phi)\lor(\delta\land\psi)}\,(\lor_e)\ x,y$$

2. Construa deduções para provar os sequentes a seguir e e indique se foi utilizada a lógica minimal, intuicionista ou clássica:

(a) $\phi \lor \psi \dashv\vdash \neg(\neg\phi \land \neg\psi)$.

Minimal

$$\dfrac{\dfrac{\phi\lor\psi \qquad \dfrac{\dfrac{[\neg\phi\land\neg\psi]^u}{\neg\phi}\,(\land_e) \qquad [\phi]^x}{\bot}\,(\neg_e) \qquad \dfrac{\dfrac{[\neg\phi\land\neg\psi]^u}{\neg\psi}\,(\land_e) \qquad [\psi]^y}{\bot}\,(\neg_e)}{\bot}\,(\lor_e)\ x,y}{\neg(\neg\phi\land\neg\psi)}\,(\neg_i)\ u$$

Clássica

$$\dfrac{\dfrac{\neg(\neg\phi\wedge\neg\psi)\qquad \dfrac{\dfrac{\dfrac{[\neg(\phi\vee\psi)]^w \qquad \dfrac{[\phi]^u}{\phi\vee\psi}\,(\vee_i)}{\bot}\,(\neg_e)}{\neg\phi}\,(\neg_i)\ u \qquad \dfrac{\dfrac{[\neg(\phi\vee\psi)]^w \qquad \dfrac{[\psi]^v}{\phi\vee\psi}\,(\vee_i)}{\bot}\,(\neg_e)}{\neg\psi}\,(\neg_i)\ v}{\neg\phi\wedge\neg\psi}\,(\wedge_i)}{\bot}\,(\neg_e)}{\phi\vee\psi}\,(\text{PBC})\ w$$

(b) $\phi\wedge\psi \dashv\vdash \neg(\neg\phi\vee\neg\psi)$.

Minimal

$$\dfrac{\dfrac{[\neg\phi\vee\neg\psi]^u \qquad \dfrac{\dfrac{\phi\wedge\psi}{\phi}\,(\wedge_e) \qquad [\neg\phi]^x}{\bot}\,(\neg_e) \qquad \dfrac{\dfrac{\phi\wedge\psi}{\psi}\,(\wedge_e) \qquad [\neg\psi]^y}{\bot}\,(\neg_e)}{\bot}\,(\vee_e)\ x,y}{\neg(\neg\phi\vee\neg\psi)}\,(\neg_i)\ u$$

Clássica

$$\dfrac{\dfrac{\neg(\neg\phi\vee\neg\psi) \qquad \dfrac{\dfrac{}{\psi\vee\neg\psi}\,((\text{LEM})) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\phi]^z \qquad [\psi]^x}{\phi\wedge\psi}\,(\wedge_i) \qquad \neg(\phi\wedge\psi)}{\bot}\,(\neg_e)}{\neg\phi}\,(\neg_i)\ z}{\neg\phi\vee\neg\psi}\,(\vee_i) \qquad \dfrac{[\neg\psi]^y}{\neg\phi\vee\neg\psi}\,(\vee_i)}{\neg\phi\vee\neg\psi}\,(\vee_e)\ x,y}{\bot}\,(\neg_e)}{\phi\wedge\psi}\,(\text{PBC})\ u$$

(c) $\varphi\to\psi \dashv\vdash (\neg\varphi)\vee\psi$

Clássica

$$\dfrac{\dfrac{}{\varphi\vee\neg\varphi}\,(\text{LEM}) \qquad \dfrac{\dfrac{[\varphi]^x \qquad \varphi\to\psi}{\psi}\,(\to_e)}{\neg\varphi\vee\psi}\,(\vee_i) \qquad \dfrac{[\neg\varphi]^y}{\neg\varphi\vee\psi}\,(\vee_i)}{\neg\varphi\vee\psi}\,(\vee_e)\ x,y$$

Intuicionista

$$\dfrac{\dfrac{(\neg\varphi)\vee\psi \qquad \dfrac{\dfrac{[\varphi]^w \qquad [\neg\varphi]^x}{\bot}\,(\neg_e)}{\psi}\,(\bot_e) \qquad [\psi]^y}{\psi}\,(\vee_e)\; x,y}{\varphi\to\psi}\,(\to_i)\; w$$

(d) $\varphi\wedge\psi \dashv\vdash \neg(\varphi\to\neg\psi)$

Intuicionista

$$\dfrac{\dfrac{\dfrac{\varphi\wedge\psi}{\psi}\,(\wedge_e) \qquad \dfrac{\dfrac{\varphi\wedge\psi}{\varphi}\,(\wedge_e) \qquad [\varphi\to\neg\psi]^x}{\neg\psi}\,(\to_e)}{\bot}\,(\neg_e)}{\neg(\varphi\to\neg\psi)}\,(\neg_i)\; x$$

Clássica

$$\dfrac{\dfrac{\dfrac{\neg(\varphi\to\neg\psi) \qquad \dfrac{\dfrac{\dfrac{[\varphi]^x \qquad [\neg\varphi]^y}{\bot}\,(\bot_e)}{\neg\psi}\,(\bot_e)}{\varphi\to\neg\psi}\,(\to_i)\; x}{\bot}\,(\neg_e)}{\varphi}\,(\text{PBC})\; y \qquad \dfrac{\dfrac{\neg(\varphi\to\neg\psi) \qquad \dfrac{[\neg\psi]^z}{\varphi\to\neg\psi}\,(\to_i)\;\emptyset}{\bot}\,(\neg_e)}{\psi}\,(\text{PBC})\; z}{\varphi\wedge\psi}\,(\wedge_i)$$

(e) $\varphi\vee\psi \dashv\vdash (\neg\varphi)\to\psi$

Clássica

$$\dfrac{\dfrac{}{\varphi\vee\neg\varphi}\,(\text{LEM}) \qquad \dfrac{\dfrac{[\neg\varphi]^x \qquad (\neg\varphi)\to\psi}{\psi}\,(\to_e)}{\varphi\vee\psi}\,(\vee_i) \qquad \dfrac{[\varphi]^y}{\varphi\vee\psi}\,(\vee_i)}{\varphi\vee\psi}\,(\vee_e)\; x,y$$

Intuicionista

$$
\dfrac{\dfrac{\varphi \lor \psi \qquad \dfrac{\dfrac{[\neg\varphi]^w \qquad [\varphi]^x}{\bot}\;(\neg_e)}{\psi}\;(\bot_e) \qquad [\psi]^y}{\psi}\;(\lor_e)\; x, y}{(\neg\varphi) \to \psi}\;(\to_i)\; w
$$

3. A lógica clássica é obtida acrescentando-se qualquer uma das seguintes regras à lógica proposicional intuicionista:

$$
\dfrac{\begin{matrix}[\neg\phi]^u \\ \vdots \\ \bot\end{matrix}}{\phi}\;(\mathrm{PBC})\; u \qquad\qquad \dfrac{}{\phi \lor \neg\phi}\;(\mathrm{LEM})
$$

$$
\dfrac{\neg\neg\phi}{\phi}\;(\neg\neg_e) \qquad\qquad \dfrac{}{((\phi \to \psi) \to \phi) \to \phi}\;(\mathrm{LP})
$$

A quarta regra (LP) é denominada Lei de Peirce. Demonstre que quaisquer três destas regras pode ser provada a partir da quarta regra restante e as regras da lógica intuicionista, ou seja:

(a) Adicione a regra (PBC) ao conjunto de regras da lógica proposicional intuicionista. Com este novo conjunto de regras prove os sequentes correspondentes à lei do terceiro excluído e à eliminação da dupla negação:

i. $\vdash \phi \lor \neg\phi$

Solução

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^v}{\phi \lor \neg\phi}\;(\lor_i) \qquad [\neg(\phi \lor \neg\phi)]^u}{\bot}\;(\neg_e)}{\neg\phi}\;(\neg_i)\; v}{\phi \lor \neg\phi}\;(\lor_i) \qquad [\neg(\phi \lor \neg\phi)]^u}{\bot}\;(\neg_e)}{\phi \lor \neg\phi}\;(\mathrm{PBC})\; u
$$

ii. $\neg\neg\phi \vdash \phi$

Solução

$$
\begin{prooftree}
\AxiomC{$\neg\neg\phi$}
\AxiomC{$[\neg\phi]^u$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{(PBC) u}
\UnaryInfC{ϕ}
\end{prooftree}
$$

iii. $\vdash ((\phi \to \psi) \to \phi) \to \phi$

Solução

$$
\begin{prooftree}
\AxiomC{$[((\phi \to \psi) \to \phi)]^x$}
\AxiomC{$[\neg\phi]^u$}
\RightLabel{$(\to_i)\ \emptyset$}
\UnaryInfC{$\neg\psi \to \neg\phi$}
\AxiomC{$[\neg\psi]^v$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\neg\phi$}
\AxiomC{$[\phi]^w$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{(PBC) v}
\UnaryInfC{ψ}
\RightLabel{$(\to_i)\ w$}
\UnaryInfC{$\phi \to \psi$}
\RightLabel{$(\to_e)$}
\BinaryInfC{ϕ}
\AxiomC{$[\neg\phi]^u$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{(PBC) u}
\UnaryInfC{ϕ}
\RightLabel{$(\to_i)\ x$}
\UnaryInfC{$((\phi \to \psi) \to \phi) \to \phi$}
\end{prooftree}
$$

(b) Adicione a regra $(\neg\neg_e)$ ao conjunto de regras da lógica proposicional intuicionista. Com este novo conjunto de regras prove:

i. $\vdash \phi \vee \neg\phi$

Solução

$$
\begin{prooftree}
\AxiomC{$[\phi]^v$}
\RightLabel{$(\vee_i)$}
\UnaryInfC{$\phi \vee \neg\phi$}
\AxiomC{$[\neg(\phi \vee \neg\phi)]^u$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ v$}
\UnaryInfC{$\neg\phi$}
\RightLabel{$(\vee_i)$}
\UnaryInfC{$\phi \vee \neg\phi$}
\AxiomC{$[\neg(\phi \vee \neg\phi)]^u$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ u$}
\UnaryInfC{$\neg\neg(\phi \vee \neg\phi)$}
\RightLabel{$(\neg\neg_e)$}
\UnaryInfC{$\phi \vee \neg\phi$}
\end{prooftree}
$$

ii. $\neg\phi \to \bot \vdash \phi$

Solução

$$
\begin{prooftree}
\AxiomC{$\neg\phi \to \bot$}
\AxiomC{$[\neg\phi]^u$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ u$}
\UnaryInfC{$\neg\neg\phi$}
\RightLabel{$(\neg\neg_e)$}
\UnaryInfC{ϕ}
\end{prooftree}
$$

iii. $\vdash ((\phi \to \psi) \to \phi) \to \phi$
Solução

$$
\begin{prooftree}
\AxiomC{$[\phi]^1$}
\AxiomC{$[\neg\phi]^y$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\bot_e)$}
\UnaryInfC{ψ}
\RightLabel{$(\to_i)\ x$}
\UnaryInfC{$\phi \to \psi$}
\AxiomC{$[(\phi \to \psi) \to \phi]^z$}
\RightLabel{$(\to_e)$}
\BinaryInfC{ϕ}
\AxiomC{$[\neg\phi]^y$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ y$}
\UnaryInfC{$\neg\neg\phi$}
\RightLabel{$(\neg\neg_e)$}
\UnaryInfC{ϕ}
\RightLabel{$(\to_i)\ z$}
\UnaryInfC{$((\phi \to \psi) \to \phi) \to \phi$}
\end{prooftree}
$$

(c) Adicione a regra (LEM) ao conjunto de regras da lógica proposicional intuicionista. Com este novo conjunto de regras prove:

i. $\neg\phi \to \bot \vdash \phi$
Solução

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(LEM)}
\UnaryInfC{$\phi \vee \neg\phi$}
\AxiomC{$[\phi]^v$}
\AxiomC{$\neg\phi \to \bot$}
\AxiomC{$[\neg\phi]^u$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\bot_e)$}
\UnaryInfC{ϕ}
\RightLabel{$(\vee_e)\ v, u$}
\TrinaryInfC{ϕ}
\end{prooftree}
$$

ii. $\neg\neg\phi \vdash \phi$
Solução

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(LEM)}
\UnaryInfC{$\phi \vee \neg\phi$}
\AxiomC{$[\phi]^v$}
\AxiomC{$\neg\neg\phi$}
\AxiomC{$[\neg\phi]^u$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\bot_e)$}
\UnaryInfC{ϕ}
\RightLabel{$(\vee_e)\ v, u$}
\TrinaryInfC{ϕ}
\end{prooftree}
$$

iii. $\vdash ((\phi \to \psi) \to \phi) \to \phi$

Solução

$$
\dfrac{
\dfrac{}{\phi \vee \neg\phi}\,(\text{LEM})
\qquad
[\phi]^z
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{[\neg\phi]^x \qquad [\phi]^y}{\bot}\,(\neg_e)
}{\psi}\,(\bot_e)
}{\phi \to \psi}\,(\to_i)\ y
\qquad
[(\phi \to \psi) \to \phi]^w
}{\phi}\,(\to_e)
}{
\dfrac{\phi}{((\phi \to \psi) \to \phi) \to \phi}\,(\to_i)\ w
}\,(\vee_e)\ x, z
$$

(d) Adicione a regra (LP) ao conjunto de regras da lógica proposicional intuicionista. Com este novo conjunto de regras prove:

i. $\neg\phi \to \bot \vdash \phi$

Solução

$$
\dfrac{
((\phi \to \neg\phi) \to \phi) \to \phi
\qquad
\dfrac{
\dfrac{
\dfrac{
\neg\phi \to \bot
\qquad
\dfrac{
\dfrac{
\dfrac{[\phi \to \neg\phi]^x \qquad [\phi]^y}{\neg\phi}\,(\to_e)
\qquad
\neg\phi \to \bot
}{\bot}\,(\to_e)
}{\neg\phi}\,(\neg_i)\ y
}{\bot}\,(\to_e)
}{\phi}\,(\bot_e)
}{(\phi \to \neg\phi) \to \phi}\,(\to_i)\ x
}{\phi}\,(\to_e)
$$

ii. $\neg\neg\phi \vdash \phi$

Solução

$$
\dfrac{
((\phi \to \neg\phi) \to \phi) \to \phi
\qquad
\dfrac{
\dfrac{
\dfrac{
\neg\neg\phi
\qquad
\dfrac{[\phi \to (\neg\phi)]^x \qquad \neg\neg\phi}{\neg\phi}\,(\text{MT})
}{\bot}\,(\neg_e)
}{\phi}\,(\bot_e)
}{(\phi \to \neg\phi) \to \phi}\,(\to_i)\ x
}{\phi}\,(\to_e)
$$

iii. $\vdash \phi \vee \neg\phi$

Solução

$$
\begin{prooftree}
\AxiomC{$(((\phi \vee \neg\phi) \to \neg\phi) \to (\phi \vee \neg\phi)) \to (\phi \vee \neg\phi)$}
\AxiomC{$[\phi]^x$}
\AxiomC{$[(\phi \vee \neg\phi) \to \neg\phi]^y$}
\AxiomC{$[\phi]^x$}
\RightLabel{$(\vee_i)$}
\UnaryInfC{$\phi \vee \neg\phi$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\neg\phi$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ x$}
\UnaryInfC{$\neg\phi$}
\RightLabel{$(\vee_i)$}
\UnaryInfC{$\phi \vee \neg\phi$}
\RightLabel{$(\to_i)\ y$}
\UnaryInfC{$(((\phi \vee \neg\phi) \to \neg\phi) \to (\phi \vee \neg\phi))$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\phi \vee \neg\phi$}
\end{prooftree}
$$

4. Construa provas para todas as variantes[1] das regras (MT) e (CP) e indique quais derivações são da lógica clássica e quais da lógica intuicionista proposicional:

$$
\frac{\pm\phi \to \pm\psi \qquad \mp\psi}{\mp\phi}\ (\mathrm{MT}_{1 \text{ e } 2}) \qquad\qquad \frac{\pm/\pm\phi \to \pm/\mp\psi}{\mp/\pm\psi \to \mp/\mp\phi}\ (\mathrm{CP}_{1,2,3 \text{ e } 4})
$$

Solução

[1] Observe que na notação deste exercício, $+\varphi$ denota a própria fórmula φ, enquanto que $-\varphi$ denota a negação de φ, ou seja, $\neg\varphi$.

Intuicionista

$$\frac{]}{}\ (\neg_e)$$

$$\begin{prooftree}
\AxiomC{$\phi \to \psi$}
\AxiomC{$[\phi]^x$}
\RightLabel{$(\to_e)$}
\BinaryInfC{ψ}
\AxiomC{$\neg\psi$}
\AxiomC{$\bot$}
\RightLabel{$(\neg_i)\ x$}
\TrinaryInfC{$\neg\phi$}
\end{prooftree}$$

Clássica

$$\begin{prooftree}
\AxiomC{$\neg\phi \to \neg\psi$}
\AxiomC{$[\neg\phi]^x$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\neg\psi$}
\AxiomC{ψ}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{(PBC) x}
\UnaryInfC{ϕ}
\end{prooftree}$$

Intuicionista

$$\begin{prooftree}
\AxiomC{$\phi \to \psi$}
\AxiomC{$[\neg\psi]^u$}
\RightLabel{(MT)}
\BinaryInfC{$\neg\phi$}
\RightLabel{$(\to_i)\ u$}
\UnaryInfC{$\neg\psi \to \neg\phi$}
\end{prooftree}$$

Clássica

$$\begin{prooftree}
\AxiomC{$\neg\phi \to \neg\psi$}
\AxiomC{$[\psi]^u$}
\RightLabel{$(\neg\neg_i)$}
\UnaryInfC{$\neg\neg\psi$}
\RightLabel{(MT)}
\BinaryInfC{$\neg\neg\phi$}
\RightLabel{$(\neg\neg_e)$}
\UnaryInfC{ϕ}
\RightLabel{$(\to_i)\ u$}
\UnaryInfC{$\psi \to \phi$}
\end{prooftree}$$

Intuicionista

$$\begin{prooftree}
\AxiomC{$\phi \to \neg\psi$}
\AxiomC{$[\phi]^x$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\neg\psi$}
\AxiomC{$[\psi]^y$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ x$}
\UnaryInfC{$\neg\phi$}
\RightLabel{$(\to_i)\ y$}
\UnaryInfC{$\psi \to \neg\phi$}
\end{prooftree}$$

Clássica

$$\begin{prooftree}
\AxiomC{$\neg\phi \to \psi$}
\AxiomC{$[\neg\phi]^x$}
\RightLabel{$(\to_e)$}
\BinaryInfC{ψ}
\AxiomC{$[\neg\psi]^y$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{(PBC) x}
\UnaryInfC{ϕ}
\RightLabel{$(\to_i)\ y$}
\UnaryInfC{$\neg\psi \to \phi$}
\end{prooftree}$$

5. Prove que não existe uma derivação intuicionista para os sequentes a seguir:

(a) $\neg\exists_x\neg\varphi \vdash \forall_x\varphi$

(b) $\neg\forall_x\neg\varphi \vdash \exists_x\varphi$

(c) $\varphi \to \psi \vdash (\neg\varphi) \vee \psi$

(a) Usam-se apenas regras minimais e assume-se a existência de uma derivação para $\neg\exists_x\neg\varphi \vdash \forall_x\varphi$:

$$
\begin{prooftree}
\AxiomC{$[\exists_x\neg\varphi]^u$}
\AxiomC{$\neg\neg\forall_x\varphi$}
\AxiomC{$[\forall_x\varphi]^w$}
\RightLabel{$(\forall_e)$}
\UnaryInfC{$\varphi(x_0)$}
\AxiomC{$[\neg\varphi(x_0)]^v$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ w$}
\UnaryInfC{$\neg\forall_x\varphi$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\exists_e)\ v$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ u$}
\UnaryInfC{$\neg\exists_x\neg\varphi$}
\RightLabel{Ass.}
\UnaryInfC{$\forall_x\varphi$}
\end{prooftree}
$$

(b) como no item anterior, assume-se a existência de uma derivação para $\neg\forall_x\neg\varphi \vdash \exists_x\varphi$, e usa-se apenas regras minimais:

$$
\begin{prooftree}
\AxiomC{$\neg\neg\exists_x\varphi$}
\AxiomC{$[\exists_x\varphi]^v$}
\AxiomC{$[\forall_x\neg\varphi]^u$}
\RightLabel{$(\forall_e)$}
\UnaryInfC{$\neg\varphi(x_0)$}
\AxiomC{$[\varphi(x_0)]^w$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\exists_e)\ w$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ v$}
\UnaryInfC{$\neg\exists_x\varphi$}
\RightLabel{$(\neg_e)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_i)\ u$}
\UnaryInfC{$\neg\forall_x\neg\varphi$}
\RightLabel{Ass.}
\UnaryInfC{$\exists_x\varphi$}
\end{prooftree}
$$

(c) como nos itens anteriores, assume-se a existência de uma derivação para $\varphi \to \psi \vdash (\neg\varphi)\vee\psi$, e usa-se apenas regras minimais. Abaixo constrói-se uma derivação para a lei do terceiro excluído:

$$
\begin{prooftree}
\AxiomC{$[\varphi]^u$}
\RightLabel{$(\to_i)\ u$}
\UnaryInfC{$\varphi \to \varphi$}
\RightLabel{Ass.}
\UnaryInfC{$(\neg\varphi) \vee \varphi$}
\end{prooftree}
$$