a. $\neg(\neg\phi \wedge \neg\psi) \vdash \phi \vee \psi$.

b. Peirce's law: $\vdash ((\phi \to \psi) \to \phi) \to \phi$.

Exercise 16. (*) *Let* Γ *be a set, and* φ *be a formula of propositional logic. Prove that if* φ *has a classical proof from the assumptions in* Γ *then* $\neg\neg\varphi$ *has an intuitionistic proof from the same assumptions. This fact is known as Glivenko's theorem (1929).*

Exercise 17. (*) *Consider the negative Gödel translation from classical propositional logic to intuitionistic propositional logic given by:*

- $\bot^n = \bot$
- $p^n = \neg\neg p$, *if* p *is a propositional variable.*
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \vee \psi)^n = \neg\neg(\varphi^n \vee \psi^n)$
- $(\varphi \to \psi)^n = \varphi^n \to \psi^n$

Prove that if $\Gamma \vdash \varphi$ *in classical propositional logic then* $\Gamma^n \vdash \varphi^n$ *in intuitionistic propositional logic.*

Exercise 18. *Prove the following sequent, the double negation of Peirce's law, in the intuitionistic propositional logic:* $\vdash \neg\neg(((\phi \to \psi) \to \phi) \to \phi)$

1.5 Semantics of the Propositional Logic

Deduction and derivation correspond to mechanical inference of *truth*. All syntactic deductive mechanisms that we have seen in the previous section can be blindly followed in order to prove that a formula of the propositional logic "holds", but in fact there was not presented a semantical counterpart of the notion of *being provable*. In this section we present the simple semantics of propositional logic.

In propositional logic the two only possible *truth-values* are *True* and *False*, denoted by brevity as T and F. No other truth-values are admissible, as it is the case in several other logical systems (e.g., truth-values as *may be true*, *probably*, *don't know*, *almost true*, *not yet, but in the future*, etc.).

Definition 9 (Truth values of atomic formula and assignments)**.** *In propositional logic the truth-values of the basic syntactic formula, that are* $\bot$*,* $\top$ *and variables in* V*, are given in the following manner:*

- *the truth-value of* $\bot$ *is* F*;*
- *the truth-value of* $\top$ *is* T*;*
- *the truth-value of a variable* v *in the set of variables* V*, is given trough a propositional assignment function from* V *to* $\{T, F\}$*. Thus, given an assignment function* $d : V \to \{T, F\}$*, the truth-value of* $v \in V$ *is given by* $d(v)$*.*

The truth-value assignment to propositional variables deserve special attention. Firstly, an assignment is necessary, because variables neither can be interpreted as true or false without having fixed an assignment. Secondly, only after one has an assignment, it is possible to decide whether (the truth-value of) a variable is either true or false. Finally, the true-value of propositional variables exclusively depends of a unique given assignment function.

Once an assignment function is given, one can determine the truth-value or semantical interpretation of non atomic propositional formulas according to the following inductive definition.

Definition 10 (Interpretation of propositional formula)**.** *Given an assignment* d *over the set of variables* V*, the truth-value or interpretation of a propositional formula* φ *is determined inductively as below:*

i. *If* $\varphi = \bot$ *or* $\varphi = \top$*, one says that* φ *is* F *or* T*, respectively;*

ii. *if* $\varphi = v \in V$*, one says that* φ *is* $d(v)$*;*

iii. if $\varphi = (\neg\psi)$, then its interpretation is given from the interpretation of ψ by the truth-table below:

ψ	$\varphi = (\neg\psi)$
T	F
F	T

iv. if $\varphi = (\psi \vee \phi)$, then its interpretation is given from the interpretations of ψ and ϕ according to the truth-table below:

ψ	ϕ	$\varphi = (\psi \vee \phi)$
T	T	T
T	F	T
F	T	T
F	F	F

v. if $\varphi = (\psi \wedge \phi)$, then its interpretation is given from the interpretations of ψ and ϕ according to the truth-table below:

ψ	ϕ	$\varphi = (\psi \wedge \phi)$
T	T	T
T	F	F
F	T	F
F	F	F

vi. if $\varphi = (\psi \to \phi)$, then its interpretation is given from the interpretations of ψ and ϕ

according to the truth-table below:

ψ	ϕ	$\varphi = (\psi \to \phi)$
T	T	T
T	F	F
F	T	T
F	F	T

According to this definition, it is possible to determine *the* truth-value of any propositional formula under a specific assignment. For instance, to determine that the formula $(v \to (\neg v))$ is false for a given assignment d for which $d(v) = T$, one can build the following *truth-table* according to the assignment of v under d and the inductive steps for the connectives $\neg$ and $\to$ of the definition:

v	$(\neg v)$	$(v \to (\neg v))$
T	F	F

Similarly, if d' is an assignment for which, $d'(v) = F$, one obtains the following *truth-table*:

v	$(\neg v)$	$(v \to (\neg v))$
F	T	T

Notice, that *the* interpretation of a formula depends on the given assignment. Also, although we are talking about *the* interpretation of a formula under a given assignment it was not proved that, given an assignment, formulas have a unique interpretation. That is done in the following lemma.

Lemma 1 (Uniqueness of interpretations)**.** *The interpretation of a propositional formula φ under a given assignment d is unique and it is either true or false.*

Proof. The proof is by induction on the structure of propositional formulas.

IB In the three possible cases the truth-value is unique: for $\bot$ false, for $\top$ true and for $v \in V$, $d(v)$ that is unique since d is functional.

IS This is done by cases.

Case $\varphi = (\neg\psi)$. By the hypothesis of induction ψ is either true or false and consequently, following the item *iii.* of the definition of interpretation of propositional formulas, the interpretation of φ is univocally given by either false or true, respectively.

Case $\varphi = (\psi \vee \phi)$. By the hypothesis of induction the truth-values of ψ and ϕ are unique and consequently, according to the item *iv.* of the definition of interpretation of propositional formulas, the truth-value of φ is unique.

Case $\varphi = (\psi \wedge \phi)$. By the hypothesis of induction the truth-values of ψ and ϕ are unique and consequently, according to the item *v.* of the definition of interpretation of propositional formulas, the truth-value of φ is unique.

Case $\varphi = (\psi \rightarrow \phi)$. By the hypothesis of induction the truth-values of ψ and ϕ are unique and consequently, according to the item *vi.* of the definition of interpretation of propositional formulas, the truth-value of φ is unique.

$\square$

It should be noticed that a formula may be interpreted both as true and false for different assignments. Uniqueness of the interpretation of a formula holds only once an assignment is fixed. Notice, for instance that the formula $(v \rightarrow (\neg v))$ can be true or false, according to the selected assignment. If it maps v to T, the formula is false and in the case that it maps v to F, the formula is true.

Whenever a formula can be interpreted as true for some assignment, it is said that the formula is satisfiable. In the other case it is said that the formula is unsatisfiable or invalid.

Definition 11 (Satisfiability and unsatisfiability). *Let φ be a propositional formula. If there exists an assignment d, such that φ is true under d, then it is said to be satisfiable. If there does not exist such an assignment, it is said that φ is unsatisfiable.*

The semantical counterpart of derivability is the notion of being a *logical consequence.*

Definition 12 (Logical consequence and validity). *Let, $\Gamma = \{\phi_1, \dots, \phi_n\}$ be a finite set of propositional formulas that can be empty, and φ be a propositional formula. Whenever for* *all assignments under which all formulas of* Γ *are true, also* φ *is true, one says that* φ *is a logical consequence of* Γ*, which is denoted as*

$$\Gamma \models \varphi$$

When Γ *is the empty set one says that* φ *is valid, which is denoted as*

$$\models \varphi$$

Notice that the notion of validity of a propositional formula φ, corresponds to the nonexistence of assignments for which φ is false. Then by simple observations of the definitions, we have the following lemma.

Lemma 2 (Satisfiability versus validity)**.**

i. Any valid formula is satisfiable.

ii. The negation of a valid formula is unsatisfiable

Proof. i. Let φ be a propositional formula such that $\models \varphi$. Then given any assignment d, φ is true under d. Thus, φ is satisfiable.

ii. Let φ be a formula such that $\models \varphi$. Then for all assignments φ is true, which implies that for all assignments $(\neg\varphi)$ is false. Then there is no possible assignment for which $(\neg\varphi)$ is true. Thus, $(\neg\varphi)$ is unsatisfiable.

□

1.6 Soundness and Completeness of the Propositional Logic

The notions of *soundness* (or *correctness*) and *completeness* are not restricted to deductive systems being also applied in several areas of computer science. For instance, we can say