

2 de agosto de 2021

Lista: Lógica Proposicional - Dedução Natural (Gabarito)

1. Prove os seguintes a seguir:

(a)  $\neg\neg\neg\phi \dashv\vdash \neg\phi$ .

**Solução**

$$\begin{array}{c}
 (\neg_e) \frac{(\neg_e) \frac{[\neg\phi]^u \quad [\phi]^v}{\perp}}{(\neg_i) u \quad \neg\neg\phi}}{(\neg_e) \neg\neg\neg\phi} \\
 (\neg_i) v \frac{\perp}{\neg\phi}
 \end{array}$$

$$\begin{array}{c}
 \frac{\neg\phi \quad [ \neg\neg\phi ]^u}{\perp} (\neg_e) \\
 \frac{\perp}{\neg\neg\neg\phi} (\neg_i) u
 \end{array}$$

(b)  $\neg\neg(\phi \rightarrow \psi) \vdash (\neg\neg\phi) \rightarrow (\neg\neg\psi)$ .

**Solução**

$$\begin{array}{c}
 \frac{(\neg_e) \frac{(\neg_e) \frac{[\phi \rightarrow \psi]^x \quad [\phi]^y}{\psi}}{(\neg_e) \psi}}{(\neg_i) y} \\
 \frac{(\neg_e) \frac{(\neg_e) \frac{[\neg\psi]^v \quad \psi}{\perp}}{(\neg_e) \neg\phi}}{(\neg_i) x} \\
 \frac{(\neg_e) \frac{(\neg_e) \frac{[\neg\neg\phi]^u \quad \neg(\phi \rightarrow \psi)}{\perp}}{(\neg_e) \neg\neg\psi}}{(\neg_i) v} \\
 \frac{\neg\neg\psi}{\neg\neg\phi \rightarrow \neg\neg\psi} (\rightarrow_i) u
 \end{array}$$

Note que o outro sentido é minimal:

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{[\psi]^w}{\phi \rightarrow \psi} (\rightarrow_i) \emptyset}{\perp} (\neg_e)}{\neg \psi} (\neg_i) w}{\perp} (\neg_e)}{\neg(\phi \rightarrow \psi)^x} \quad \frac{\frac{\frac{\frac{[\phi]^y \quad [\neg\phi]^z}{\perp} (\neg_e)}{\psi} (\perp_e)}{\phi \rightarrow \psi} (\rightarrow_i) y}{\perp} (\neg_e)}{\neg(\phi \rightarrow \psi)^x} \\
 \frac{\neg\neg\phi \rightarrow \neg\neg\psi}{\perp} \quad \frac{\neg\neg\phi}{\neg\neg\psi} \\
 \frac{\perp}{\neg\neg(\phi \rightarrow \psi)} (\neg_e) x
 \end{array}$$

(c)  $\neg\neg(\phi \wedge \psi) \dashv\vdash \neg\neg\phi \wedge \neg\neg\psi$ .

**Solução**

$$\begin{array}{c}
 \frac{\frac{\frac{[\neg\phi]^u}{\perp} (\neg_e)}{\neg(\phi \wedge \psi)} (\neg_e)}{\neg\neg(\phi \wedge \psi)} \quad \frac{\frac{\frac{[\phi \wedge \psi]^v}{\phi} (\wedge_e)}{\perp} (\neg_e)}{\neg(\phi \wedge \psi)} (\neg_i) v}{\perp} (\neg_e)}{\neg\neg\phi} (\neg_i) u \\
 \frac{\frac{\frac{[\neg\psi]^x}{\perp} (\neg_e)}{\neg(\phi \wedge \psi)} (\neg_e)}{\neg\neg(\phi \wedge \psi)} \quad \frac{\frac{\frac{[\phi \wedge \psi]^y}{\psi} (\wedge_e)}{\perp} (\neg_e)}{\neg(\phi \wedge \psi)} (\neg_i) y}{\perp} (\neg_e)}{\neg\neg\psi} (\neg_i) x \\
 \frac{\neg\neg\phi \wedge \neg\neg\psi}{\neg\neg\psi} (\wedge_i)
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{\frac{(\wedge_i) \frac{[\phi]^x \quad [\psi]^y}{\phi \wedge \psi}}{\perp} (\neg_e)}{(\wedge_e) \frac{(\neg\neg\phi) \wedge (\neg\neg\psi)}{\neg\neg\phi}} (\neg_i) x}{\perp} (\neg_e)}{(\wedge_e) \frac{(\neg\neg\phi) \wedge (\neg\neg\psi)}{\neg\neg\psi}} (\neg_i) y} \\
 \frac{\perp}{\neg\neg(\phi \wedge \psi)} (\neg_e) z
 \end{array}$$

(d)  $\neg(\phi \vee \psi) \dashv\vdash \neg\phi \wedge \neg\psi$ .

**Solução**

$$\begin{array}{c}
 \frac{\frac{\frac{(\neg_e) \neg(\phi \vee \psi)}{\perp} (\neg_e)}{(\neg_i) u} \quad \frac{\frac{(\vee_i) \frac{[\phi]^u}{\phi \vee \psi}}{\perp} (\neg_e)}{\neg\phi} (\neg_e)}{\perp} (\neg_e)}{\neg\phi \wedge \neg\psi} \\
 \frac{\frac{\frac{(\neg_e) \neg(\phi \vee \psi)}{\perp} (\neg_e)}{(\neg_i) v} \quad \frac{\frac{(\vee_i) \frac{[\psi]^v}{\phi \vee \psi}}{\perp} (\neg_e)}{\neg\psi} (\neg_e)}{\perp} (\neg_e)}{\neg\psi} (\neg_e)}{\neg\phi \wedge \neg\psi} (\wedge_i)
 \end{array}$$

$$\begin{array}{c}
\frac{(\wedge_e) \frac{(\neg\phi \wedge \neg\psi)}{\neg\phi} \quad [ \phi ]^x \quad \frac{(\neg\phi \wedge \neg\psi) (\wedge_e)}{\neg\psi} \quad [ \psi ]^y}{\perp} \quad (\neg_e) \\
\frac{\perp}{\perp} \quad (\vee_e) \ x, y \\
\frac{\perp}{\neg(\phi \vee \psi)} \quad (\neg_i) \ u
\end{array}$$

(e)  $(\phi \wedge \psi) \wedge \varphi \dashv\vdash \phi \wedge (\psi \wedge \varphi)$ .

**Solução**

$$\begin{array}{c}
\frac{(\wedge_e) \frac{(\phi \wedge \psi) \wedge \varphi}{(\phi \wedge \psi)} \quad (\wedge_e) \frac{(\phi \wedge \psi) \wedge \varphi}{\phi \wedge \psi} \quad (\wedge_e) \frac{(\phi \wedge \psi) \wedge \varphi}{\psi} \quad (\wedge_e) \frac{(\phi \wedge \psi) \wedge \varphi}{\varphi}}{\phi \quad (\psi \wedge \varphi)} \quad (\wedge_i) \\
\frac{\phi \quad (\psi \wedge \varphi)}{\phi \wedge (\psi \wedge \varphi)} \quad (\wedge_i)
\end{array}$$

$$\begin{array}{c}
\frac{(\wedge_e) \frac{\phi \wedge (\psi \wedge \varphi)}{\phi} \quad \frac{\phi \wedge (\psi \wedge \varphi)}{(\psi \wedge \varphi)} (\wedge_e) \quad \frac{\phi \wedge (\psi \wedge \varphi)}{(\psi \wedge \varphi)} (\wedge_e)}{(\phi \wedge \psi) \quad \varphi} \quad (\wedge_e) \\
\frac{(\phi \wedge \psi) \quad \varphi}{(\phi \wedge \psi) \wedge \varphi} \quad (\wedge_i)
\end{array}$$

(f)  $(\phi \vee \psi) \vee \varphi \dashv\vdash \phi \vee (\psi \vee \varphi)$ .

**Solução**

$$\begin{array}{c}
\frac{(\phi \vee \psi) \vee \varphi \quad \frac{[(\phi \vee \psi)]^x \quad \frac{[\phi]^z}{\phi \vee (\psi \vee \varphi)} (V_i) \quad \frac{[\psi]^w}{(\psi \vee \varphi)} (V_i)}{\phi \vee (\psi \vee \varphi)} (V_e) \ z, w \quad \frac{[\varphi]^y}{(\psi \vee \varphi)} (V_i)}{\phi \vee (\psi \vee \varphi)} (V_e) \ x, y
\end{array}$$

$$\frac{\phi \vee (\psi \vee \varphi) \quad \frac{\frac{[\phi]^x}{(\phi \vee \psi)} (V_i)}{(\phi \vee \psi) \vee \varphi} (V_i) \quad \frac{\frac{[(\psi \vee \varphi)]^y}{(\phi \vee \psi) \vee \varphi} (V_i) \quad \frac{\frac{[\psi]^u}{(\phi \vee \psi)} (V_i)}{(\phi \vee \psi) \vee \varphi} (V_i) \quad \frac{[\varphi]^v}{(\phi \vee \psi) \vee \varphi} (V_e) u, v}{(\phi \vee \psi) \vee \varphi} (V_e) x, y}{(\phi \vee \psi) \vee \varphi} (V_e) x, y$$

(g)  $\phi \rightarrow \psi \vdash \delta \vee \phi \rightarrow \delta \vee \psi$

**Solução**

$$\frac{\frac{\phi \rightarrow \psi \quad [\phi]^x}{\psi} (\rightarrow_e) \quad \frac{[\delta]^y}{\psi \vee \delta} (V_i)}{\frac{[\delta \vee \phi]^z \quad \psi \vee \delta}{\psi \vee \delta} (V_e) x, y} \quad \frac{\psi \vee \delta}{\delta \vee \phi \rightarrow \delta \vee \psi} (\rightarrow_i) z$$

(h)  $(\delta \wedge \phi) \vee (\delta \wedge \psi) \dashv\vdash \delta \wedge (\phi \vee \psi)$  (Distributividade)

**Solução**

$$\frac{\frac{(\delta \wedge \phi) \vee (\delta \wedge \psi) \quad \frac{[\delta \wedge \phi]^u}{\delta} (\wedge_e) \quad \frac{[\delta \wedge \psi]^v}{\delta} (\wedge_e)}{\delta} (V_e) u, v \quad \frac{\frac{(\delta \wedge \phi) \vee (\delta \wedge \psi) \quad \frac{[\delta \wedge \phi]^x}{\phi \vee \psi} (\wedge_e) \quad \frac{[\delta \wedge \psi]^t}{\phi \vee \psi} (\wedge_e)}{\phi \vee \psi} (\wedge_i)}{\delta \wedge (\phi \vee \psi)} (V_e) x, y}{\delta \wedge (\phi \vee \psi)} (\wedge_e) x, y$$

$$\frac{\frac{\delta \wedge (\phi \vee \psi)}{\delta} (\wedge_e) \quad [\phi]^x (\wedge_i) \quad \frac{\delta \wedge (\phi \vee \psi)}{\delta} (\wedge_e) \quad [\psi]^y (\wedge_i)}{\frac{\delta \wedge (\phi \vee \psi)}{\phi \vee \psi} (\wedge_e) \quad \frac{\delta \wedge \phi}{(\delta \wedge \phi) \vee (\delta \wedge \psi)} (V_i) \quad \frac{\delta \wedge \psi}{(\delta \wedge \phi) \vee (\delta \wedge \psi)} (V_i)}{\delta \wedge (\phi \vee \psi)} (V_e) x, y$$

(i)  $(\neg\neg\phi) \rightarrow (\neg\neg\psi) \vdash \neg\neg(\phi \rightarrow \psi)$ .

## Solução

$$\begin{array}{c}
 \frac{\frac{[\neg(\phi \rightarrow \psi)]^x}{\perp} \quad \frac{[\psi]^w}{\phi \rightarrow \psi} (\rightarrow_i) \emptyset}{\neg\psi} (\neg_e) w \\
 \hline
 \frac{\frac{[\neg(\phi \rightarrow \psi)]^x}{\perp} \quad \frac{[\phi]^y \quad [\neg\phi]^z}{\perp} (\neg_e)}{\psi} (\perp_e) y \\
 \frac{\frac{[\neg(\phi \rightarrow \psi)]^x}{\phi \rightarrow \psi} (\rightarrow_i) y}{\perp} (\neg_e) z \\
 \frac{\neg\neg\phi \rightarrow \neg\neg\psi}{\neg\neg\psi} (\rightarrow_e) \\
 \hline
 \frac{\perp}{\neg\neg(\phi \rightarrow \psi)} (\neg_e) x
 \end{array}$$