deduction rules and each of the assumptions:

a. $\neg\exists_x\neg\varphi \to \forall_x\varphi$ *and*

b. $\neg\forall_x\neg\varphi \to \exists_x\varphi$.

Hint: you can choose the variable x as any variable that does not occurs in φ. Thus, the application of rule $(\exists_e)$ over the existential formula $\exists_x\varphi$ has as witness assumption $[\varphi[x/x_0]]^w$ that has no occurrences of x_0.

In Exercise 24 we prove that there are intuitionistic derivations for $\neg\exists_x\varphi \dashv\vdash \forall_x\neg\varphi$. Also, in Example 12 we give an intuitionistic derivation for $\exists_x\neg\varphi \vdash \neg\forall_x\varphi$. Indeed, one can obtain minimal derivations for these three sequents.

Exercise 29. *To complete $\neg\forall_x\varphi \dashv\vdash \exists_x\neg\varphi$ (see Example 12), prove that $\neg\forall_x\varphi \vdash \exists_x\neg\varphi$.*

2.4 Semantics of the Predicate Logic

As done for the propositional logic in Chapter 1, here we present the standard semantics of first-order classical logic. The semantics of the predicate logic is not a direct extension of the one of propositional logic. Although this is not surprising, since the predicate logic has a richer language, there are some interesting points concerning the differences between propositional and predicate semantics that will be examined in this section. In fact, while a propositional formula has only finitely many interpretations, a predicate formula can have infinitely many ones.

We start with an example: let p be a unary predicate symbol, and consider the formula $\forall_x p(x)$. The variable x ranges over a domain, say the set of natural numbers $\mathbb{N}$. Is this formula true or false? Certainly, it depends on how the predicate symbol p is interpreted. If one interprets $p(x)$ as "x is a prime number", then it is false, but if $p(x)$ means that "x is a natural number" then it is true. Observe that the interpretation depends on the chosen domain, and hence the latter interpretation of p will be false over the domain of integers $\mathbb{Z}$.

This situation is similar in the propositional logic: according to the interpretation, some formulas can be either true or false. So what do we need to determine the truth value of a predicate formula? First of all, we need a domain of concrete individuals, i.e. a non-empty set $\mathtt{D}$ that represents known individuals (e.g. numbers, people, organisms, etc.). Function symbols (and constants) are associated to functions in the so called *structures*:

Definition 22 (Structure). *A structure of a first-order language L over the set $S = (\mathbb{F}, \mathbb{P})$, also called an S-structure, is a pair $\langle \mathtt{D}, \mathtt{m} \rangle$, where $\mathtt{D}$ is a non-empty set and $\mathtt{m}$ is a map defined as follows:*

1. *if f is a function symbol of arity $n \geq 0$, then $\mathtt{m}(f)$ is a function from $\mathtt{D}^n$ to $\mathtt{D}$. A function from $\mathtt{D}^0$ to $\mathtt{D}$ is simply an element of $\mathtt{D}$.*

2. *if p is a predicate symbol of arity $n > 0$, then $\mathtt{m}(p)$ is a subset of $\mathtt{D}^n$.*

Intuitively, the set $\mathtt{m}(p)$ contains the tuples of elements that satisfy the predicate p. As an example, consider the formula $q(a)$, where a is a constant, and the structure $\langle \{0, 1\}, \mathtt{m} \rangle$, where $\mathtt{m}(a) = 0$ and $\mathtt{m}(q) = \{0\}$. The formula $q(a)$ is true in this structure because the set $\mathtt{m}(q)$ contains the element 0, the image of the constant a by the function $\mathtt{m}$. But $q(a)$ would be false in other structures; for instance, it is false in the structure $\langle \{0, 1\}, \mathtt{m}' \rangle$, where $\mathtt{m}'(a) = 0$ and $\mathtt{m}'(q) = \emptyset$.

If a formula contains (free) variables, such as the formula $q(x)$, then a special mechanism is needed to interpret variables. Variables are associated to elements of the domain $\mathtt{D}$ through *assignments*, that are functions from the set of variables $\mathbb{V}$ to the domain $\mathtt{D}$. So, if d is an assignment such that $d(x) = 0$ then $q(x)$ is true in the structure $\langle \{0, 1\}, \mathtt{m} \rangle$ above, and if $d'(x) = 1$ then $q(x)$ is false.

Definition 23 (Interpretation of terms). *An interpretation I is a pair $\langle \langle \mathtt{D}, \mathtt{m} \rangle, d \rangle$ containing a structure and an assignment. Given an interpretation I and a term t, the interpretation of t by I, written t^I, is inductively defined as follows:*

1. *For each variable x, $x^I = d(x)$;*

2. *For each function symbol f with arity $n \geq 0$, $f(t_1, \ldots, t_n)^I = \mathtt{m}(f)(t_1^I, \ldots, t_n^I)$.*

Thus, based on the interpretations of terms, the semantics of predicate formulas concerns the truth value of a formula that can be either T (true) or F (false). This notion is formalized in the following definition.

Definition 24 (Interpretation of Formulas). *The truth value of a predicate formula φ according to a given interpretation of terms $I = \langle\langle \mathtt{D}, \mathtt{m}\rangle, d\rangle$, denoted as φ^I, is inductively defined as:*

1. $\bot^I = F$ *and* $\top^I = T$;

2. $p(t_1, \ldots, t_n)^I = \begin{cases} T, & \text{if } (t_1^I, \ldots, t_n^I) \in \mathtt{m}(p), \\ F, & \text{if } (t_1^I, \ldots, t_n^I) \notin \mathtt{m}(p); \end{cases}$

3. $(\neg\psi)^I = \begin{cases} T, & \text{if } \psi^I = F, \\ F, & \text{if } \psi^I = T; \end{cases}$

4. $(\psi \wedge \gamma)^I = \begin{cases} T, & \text{if } \psi^I = T \text{ and } \gamma^I = T, \\ F, & \text{otherwise}; \end{cases}$

5. $(\psi \vee \gamma)^I = \begin{cases} T, & \text{if } \psi^I = T \text{ or } \gamma^I = T, \\ F, & \text{otherwise}; \end{cases}$

6. $(\psi \to \gamma)^I = \begin{cases} F, & \text{if } \psi^I = T \text{ and } \gamma^I = F, \\ T, & \text{otherwise}; \end{cases}$

7. $(\forall_x \psi)^I = \begin{cases} T, & \text{if } \psi^{I\frac{x}{a}} = T \text{ for every } a \in \mathtt{D}, \\ F, & \text{otherwise}; \end{cases}$

8. $(\exists_x \psi)^I = \begin{cases} T, & \text{if } \psi^{I\frac{x}{a}} = T \text{ for at least one } a \in \mathtt{D}, \\ F, & \text{otherwise.} \end{cases}$

where $I\frac{x}{a}$ denotes the interpretation I modifying its assignment d, in such a way that it maps x to a, and any other variable y to $d(y)$.

Definition 25 (Models). *An interpretation I is said to be a model of φ if $\varphi^I = T$. We write $I \models \varphi$ to denote that I is a model of φ.*

The notion of Model is extended to sets of formulas in a straightforward manner: If Γ *is a set of predicate formulas then* I *is a model of* Γ*, denoted by* $I \models \Gamma$*, whenever* I *is a model of each formula in* Γ*.*

Example 14. *Let* I *be an interpretation with domain* $\mathbb{N}$ *and* $\mathtt{m}(p) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$*. Then* I *is a model of* $\forall_x \exists_y\, p(x,y)$*, denoted as* $I \models \forall_x \exists_y\, p(x,y)$*, because for every natural* x *one can find another natural* y *bigger than* x*. With similar arguments, one can conclude that* I *is not a model of* $\exists_x \forall_y\, p(x,y)$*.*

Definition 26 (Satisfiability)**.** *Let* φ *be a predicate formula. If* φ *has a model then it is said to be satisfiable; otherwise, it is unsatisfiable. This notion is also extended to sets of formulas:* Γ *is satisfiable if and only if there exists an interpretation* I *such that for all* $\varphi \in \Gamma$*,* $I \models \varphi$*.*

Definition 27 (Logical consequence and Validity)**.** *Let* $\Gamma = \{\phi_1, \ldots, \phi_n\}$ *be a finite set of predicate formulas, and* φ *a predicate formula. We say that* φ *is a logical consequence of* Γ*, denoted as* $\Gamma \models \varphi$*, if every model of* Γ *is also a model of* φ*, i.e.* $I \models \Gamma$ *implies* $I \models \varphi$*, for every interpretation* I*. When* Γ *is empty then* φ *is said to be valid, which is denoted as* $\models \varphi$*.*

Example 15. *We claim that* $\forall_x(p(x) \to q(x)) \models (\forall_x p(x)) \to (\forall_x q(x))$*. In fact, let* $I = \langle\langle \mathtt{D}, \mathtt{m}\rangle, d\rangle$ *be a model of* $\forall_x(p(x) \to q(x))$*, i.e.,* $I \models \forall_x(p(x) \to q(x))$*. If there exists an element in the domain of* I *that does not satisfy the predicate* p *then* $\forall_x p(x)$ *is false in* I *and hence,* $(\forall_x(p(x)) \to (\forall_x q(x))$ *would be true in* I*. Otherwise,* $I \models \forall_x p(x)$*, and hence* $I\frac{x}{a} \models p(x)$*, for all* $a \in \mathtt{D}$*. Since* $I \models \forall_x(p(x) \to q(x))$*, we conclude that* $I\frac{x}{a} \models q(x)$*, for all* $a \in \mathtt{D}$*. Therefore,* $I \models \forall_x q(x)$*.*

The study of models can be justified by the fact that validity in a model is an invariant of provability in the sense that a sequent is provable exactly when all its interpretations are also models. This suggests a way to prove when a sequent is not provable: it is enough to find an interpretation that is not a model of the sequent. In the next section, we formalize this for the predicate logic.